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Stability analysis of stratified shear flows with a monotonic velocity profile without inflection points

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The two-layer model of a stably stratified medium has been used to investigate the stability of flows without inflection points on the profile of the velocity $V_x = u(y)$, which is monotonically increasing from zero at the bottom $(y = 0)$ to its maximum value U_0 (when $y \to \infty$). It is shown that in the case of flows of a general form (in which $u''(y) < 0$ everywhere) an instability sets in for an arbitrarily small density difference; furthermore, perturbations of all scales build up simultaneously. With an enhancement of stratification, the real part c_r of the phase velocity of unstable perturbations increases. The upper boundary of the instability domain is determined by the fact that at a certain stratification level (a particular one for perturbations of each scale), c_r reaches U_0, the perturbation is no longer in phase resonance with the flow and turns into a neutral oscillation of the medium.

For flows of a special kind, having points of zero curvature (where $u'' = 0$) on the velocity profile (but having no inflection points as before), the influence of neutral modes, associated with these points, on the formation of the instability domain configuration is analysed, and an interpretation of this influence is given in terms of the resonance and non-resonance contributions to shear flow instability.

1. Introduction

In various problems of hydrodynamics, atmospheric and ocean physics, astrophysics, and other areas of knowledge, one encounters the stability of shear flows of a stratified medium. This topic has been addressed in a large body of research and has an extensive literature (see, for example Dikii 1976; Drazin & Reid 1981; Turner 1973); however, the overall picture still remains patchy, and many of the key questions are open.

Let us turn to the simpler and relatively well-studied case of plane-parallel flows. At the present time there are two main approaches to solving the problem of their stability. One of them is based on investigation (analytical and/or numerical) of flows with specific velocity and density profiles, $V_x = u(y)$ and $\rho = \rho_0(y)$, in particular, on seeking and analysis of exactly solvable models (e.g. Thorpe 1969). Within the framework of this approach, a representative series of model flows has been studied (see, for example, Drazin & Howard 1966; Hazel 1972; Turner 1973), and a number of interesting and important results obtained. However, the extent to which these results are general, i.e. their independence of the details of flow structure, falls far short of being always clear. Another way of looking at instability has its origins in a qualitative analysis of differential equations describing the flow, in particular, of the Taylor–Goldstein equation. It is focused on obtaining results that hold for an arbitrary flow or, at least, for a wide class of flows. Amongst these, results of great importance should be mentioned such as the Rayleigh and Fjørtoft theorems for homogeneous flows, as well as the Miles theorem and the Howard semicircle theorem and its extensions for stratified flows (see, for example, Dikii 1976; Drazin & Reid 1981; Kozyrev & Stepanyants 1991; Redekopp 2001).

This paper follows mainly the latter approach. Its objective is to investigate the stability of flows without inflection points on the velocity profile in a stably stratified medium. In homogeneous media, such flows are stable by the Rayleigh theorem so that we shall study the consequences of invalidating this theorem by including stratification.† For this purpose, we consider in the semispace $y \geqslant 0$ plane-parallel flows of incompressible fluid in a gravity field g. The flow velocity $u(y)$ increases monotonically from zero when $y = 0$ to U_0 when $y \to \infty$. We confine ourselves to analysing those flows in which the characteristic scale l of density variation is small compared to the scale L of velocity variation. Such a relationship of the scales occurs naturally in fluids with high Prandtl number, but it can be also due to another reason, e.g. by initial and/or boundary conditions. It is typical, in particular, of flows in the pycnocline, flows containing the interface between two media, as well as of a number of other flows, such as the flow of a gas accreted on the photosphere of a star, which has recently attracted considerable interest (from the standpoint of hydrodynamic stability; e.g. Rosner *et al.* 2002; Alexakis, Young & Rosner 2002). In this paper we consider the limiting case $l = 0$ when the medium consists of two homogeneous layers of different density, and the density profile has the form of a 'step':

$$\rho_0(y) = \begin{cases} \rho_1, & 0 \leqslant y < y_N, \\ \rho_2 < \rho_1, & y_N < y < \infty, \end{cases} \qquad N^2(y) \equiv -g\,\frac{\mathrm{d}}{\mathrm{d}y}\ln\rho_0 = J\delta(y - y_N), \tag{1.1}$$

where $N^2(y)$ is the squared Brunt–Väisälä frequency, J is the Richardson parameter (Richardson global number), and $\delta(z)$ is the Dirac delta function. Hereafter, we shall use dimensionless quantities scaled by U_0, L, and ρ_1.

Following the standard method for investigation of (spectral) stability in the linear approximation (e.g. Dikii 1976; Drazin & Reid 1981), we expand a perturbation in the Fourier integral in terms of the streamwise wavenumber k and then, for each k and J, we seek eigenvalues (of complex phase velocity c) for the Taylor–Goldstein equation (2.1). It is easy to show (see § 3.2) that this problem has a continuous spectrum $J = J_c(k)$ of neutral oscillations with $c \geqslant 1$, where $J_c(k)$ is a monotonically increasing function of both c and k. In accordance with the Howard semicircle theorem, unstable oscillations should have $|c| < 1$.

In the general case, the flow belonging to the class under consideration has a velocity profile such as $u'' = \mathrm{d}^2u/\mathrm{d}y^2 < 0$ when $0 < y < \infty$. The principal result of this paper is that for all such flows the instability domain on the $(k,\ J)$-plane has the universal configuration: it is in the band $0 < J < J_1(k)$ where $J = J_1(k)$ is the dispersion relation for neutral oscillations with phase velocity $c = 1$. The picture of instability is as follows. Every flow becomes unstable at an arbitrarily small J (i.e. with an arbitrarily small density difference), and disturbances of all scales, from $k = 0$ to $k = \infty$, lose stability at once. When $J \to +0$, the real part c_r of the phase velocity of the unstable perturbation tends to $u_N = u(y_N)$ at any k. With increasing J, c_r increases

† The result of a similar invalidation of the Fjørtoft theorem has been demonstrated by Thorpe (1969) who has shown that the flow $u = U_0 \sinh(y/L)$ is unstable in a stably stratified medium.

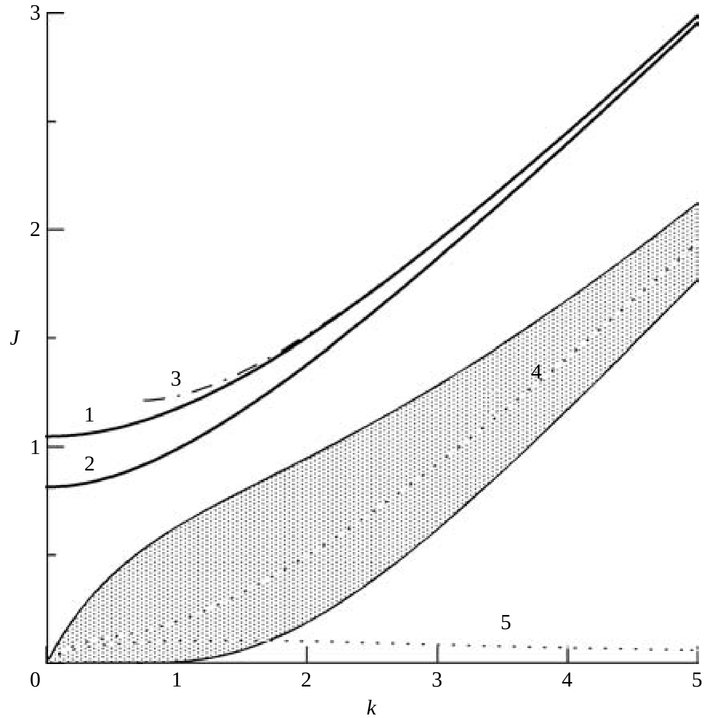

FIGURE 1. Instability domains for flows without inflection points on the velocity profile. The instability domain of the flow (1.2) is shaded. Curve 1, the upper boundary $J = J_1(k)$ of the instability domain for the flow $u = \tanh y$; 2, as curve 1 but, for $u = 1 - e^{-y}$; 3, the WKB approximation of curve 1; 4, the line at which the growth rate in the flow (1.2) takes its maximum in J; 5, as curve 4 but, in the flow $u = \tanh y$.

but the disturbances of all scales remain unstable as before. And only when c_r reaches unity (a maximum flow velocity) at some, individual for every k, value of $J = J_1(k)$, does the perturbation of a corresponding scale cease to be in phase resonance with the flow and transforms from an unstable to a neutral oscillation of the medium.

As an illustration, we have calculated numerically the stability of the flows $u = \tanh y$ and $u = 1 - e^{-y}$ for $0 < k < 10$ and have checked that the instability domain does indeed take up the whole band $0 < J < J_1(k)$ (which we call hereafter the Band), and that c_r does vary with J in the manner described above. In figure 1, the curves $J = J_1(k)$ for these flows are marked respectively 1 and 2.

As one more example, it is natural to consider the flow with the simplest piecewise linear velocity profile

$$u(y) = \begin{cases} y, & 0 \leqslant y \leqslant 1, \\ 1, & y \geqslant 1, \end{cases} \tag{1.2}$$

the stability problem for which can be solved analytically (see §5). In this model $u'' = -\delta(y-1) \leqslant 0$, and it clearly belongs to the chosen class of flows but the domain of its instability on the (k, J)-plane (shaded in figure 1), although shown inside the Band (see figure 8), has quite a different configuration. As an analysis shows, such a difference is due to the fact that nearly every point on the velocity profile (1.2) is a null-curvature point (NCP) at which $u'' = 0$, and that each NCP can create a little 'stability island' within the Band and therefore reduces the instability domain.

The paper is organized as follows. In § 2 we formulate the statement of the problem and basic equations. In § 3 the stability of flows in the general case (when $u'' < 0$ everywhere) is analysed, and § 4 is devoted to the study of changes in spectrum and stability caused by the NCP. The flow with a piecewise linear velocity profile and its place in the overall picture are considered in § 5. Results are discussed in § 6. Some of ancillary calculations are presented in the Appendices.

2. Statement of the problem and basic equations

Let us consider a steady-state plane-parallel flow $u(y)$,

$$u(0) = 0, \quad u(\infty) = 1, \quad u'(0) = 1; \quad u'(y) \geqslant 0, \quad u''(y) \leqslant 0,$$

of ideal stably stratified fluid, and two-dimensional perturbations to its background which we shall describe by a stream function ψ: $V_x = u(y) + \partial\psi/\partial y$, $V_y = -\partial\psi/\partial x$. Within the Boussinesq approximation, the linearized stream function, $\psi = g(y)\mathrm{e}^{-\mathrm{i}\omega t + \mathrm{i}kx}$, satisfies the Taylor–Goldstein equation (Drazin & Reid 1981)

$$\frac{\mathrm{d}^2 g}{\mathrm{d}y^2} + \left[\frac{N^2(y)}{(u-c)^2} - \frac{u''}{u-c} - k^2\right] g = 0; \quad g(0) = 0, \quad |g(\infty)| < \infty, \tag{2.1}$$

which in a two-layer medium (1.1) reduces to the Rayleigh equation (Drazin & Reid 1981)

$$\frac{\mathrm{d}^2 g_\pm}{\mathrm{d}y^2} - \left(\frac{u''}{u-c} + k^2\right) g_\pm = 0; \quad g_-(0) = 0, \quad |g_+(\infty)| < \infty, \tag{2.2}$$

supplemented by continuity conditions for pressure and normal velocity at the interface:

$$g_+(y_N) = g_-(y_N), \quad J = (u_N - c)^2 \left(\frac{g'_-}{g_-} - \frac{g'_+}{g_+}\right)_{y=y_N}. \tag{2.3}$$

Here the subscripts + and − correspond to the solutions for $y > y_N$ and $y < y_N$, respectively, and $c = \omega/k$. Also, we introduce the functions $F(y)$ and $G(y)$ that are the solutions of the Rayleigh equation for the entire real semiaxis $0 \leqslant y < \infty$ when $c_i = \operatorname{Im} c > 0$. They are specified by the equalities

$$G(y) = g_+(y) \quad \text{when} \quad y \geqslant y_N; \quad F(y) = G(y)\int_0^y \frac{\mathrm{d}y_1}{G^2(y_1)}; \quad F'G - FG' = 1, \tag{2.4}$$

and can be analytically continued to $c_i \leqslant 0$. If somewhere along the path of integration $G = 0$, this point should be bypassed in a complex plane. Note that $G(y)$ is a continuation of $g_+(y)$ and $F(y)$ is a continuation of $g_-(y)$ (up to normalization†) onto the whole flow domain, $0 \leqslant y < \infty$. With F and G, the second formula in (2.3) can be written as

$$J = (u_N - c)^2 \left(\frac{F'}{F} - \frac{G'}{G}\right)_{y=y_N}. \tag{2.5}$$

† $F(0) = 0$ for every k such that $G(0) \neq 0$; in this case $g_-(y)/F(y) = \text{const}$ when $y \leqslant y_N$.

Sometimes, instead of $g(y)$, it is more convenient to use the function $f(y)$: $g = (u-c)\, f\, \mathrm{e}^{-ky}$, and†

$$\left.\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}y}\left[(u-c)^2\frac{\mathrm{d}f_\pm}{\mathrm{d}y}\right] = 2k(u-c)\frac{\mathrm{d}}{\mathrm{d}y}[(u-c)f_\pm]; \quad f_-(0)=0, \quad f_+(\infty)=1;\\ &f_+(y_N) = f_-(y_N), \quad J = (u_N-c)^2\left(\frac{f'_-}{f_-} - \frac{f'_+}{f_+}\right)_{y=y_N}. \end{aligned}\right\} \tag{2.6}$$

3. Flow stability in the general case

In this section we investigate the stability of the flow with a sufficiently arbitrary velocity profile $u(y)$. It will be assumed that u'' is strictly negative for any finite $y > 0$ and that the density jump lies neither too close to or too far from the bottom, i.e. both u_N and $(1-u_N)$ are of the order of unity. First we must point out that for every k, the instability domain is bounded in J. Indeed, by the Rayleigh theorem, the functions F and G have no zeros when $0 < y < \infty$. Moreover, in accordance with the Howard semicircle theorem, in the case of unstable modes $|c| \leqslant 1$. Therefore the right-hand side of the matching condition (2.5) is bounded above, and this implies boundedness in J of the instability domain. Our aim is to determine its boundaries.

It is convenient to display the oscillation spectrum on a (k, J)-diagram. The coordinate axes are natural boundaries of the instability domain because oscillations (if any) with $k=0$ or $J=0$ are clearly neutral. In § 3.1 we calculate the oscillation spectrum for long waves ($k \ll 1$) and show that some of these oscillations are unstable. Then, in § 3.2, we find the upper boundary of the instability domain – it turns out that this is the curve $J = J_1(k)$ corresponding to neutral oscillations propagating with the phase velocity $c=1$ – and show that when $J > 0$ there are no neutral oscillations with $c < 1$; hence, there are no another stability boundaries. Thus we prove that in a flow of a general form the instability domain is indeed the band $0 < J < J_1(k)$.

3.1. *Long-wavelength perturbations*

When $k \ll 1$ it is convenient to solve the problem as formulated in (2.6) using the method developed by Drazin & Howard (1962). Straightforward but rather unwieldy calculations (see Appendix A) lead to the dispersion equation

$$\begin{aligned} &\left[J\int_0^{y_N}\frac{\mathrm{d}y}{(u-c)^2} - 1\right]\left\{1 - k\int_{y_N}^{\infty}\mathrm{d}y_1\left[1-\left(\frac{1-c}{u_1-c}\right)^2\right]\right.\\ &\qquad\left. + k^2\int_{y_N}^{\infty}\frac{\mathrm{d}y_1}{(u_1-c)^2}\int_{y_1}^{\infty}\mathrm{d}y_2[(u_2-c)^2-(1-c)^2]\right\}\\ &\qquad - Jk^2\int_0^{y_N}\mathrm{d}y(u-c)^2\left[\int_0^{y}\frac{\mathrm{d}y_1}{(u_1-c)^2}\right]^2\\ &\quad = k\left\{(1-c)^2 + k\int_{y_N}^{\infty}\mathrm{d}y_1[(u_1-c)^2-(1-c)^2]\right\}\int_0^{y_N}\frac{\mathrm{d}y}{(u-c)^2} + O(k^3), \end{aligned} \tag{3.1}$$

where the notation $u_m = u(y_m)$ is used. The pole $u(y)=c$, by the Landau rule (e.g. Dikii 1976), should be bypassed below because $u'(y) > 0$.

† The boundary condition for $y \to \infty$ corresponds to a specific normalization of $f(y)$ and $g(y)$.

For $k = 0$ we obtain

$$J \int_0^{y_N} \frac{\mathrm{d}y}{(u-c)^2} = 1, \tag{3.2}$$

so that a neutral oscillation with $c = c_0(J) > u_N$, corresponds to each value of the Richardson parameter $J > 0$, and c_0 increases with an increase of J. When

$$J > J_1(0) = \left[\int_0^{y_N} \frac{\mathrm{d}y}{(1-u)^2} \right]^{-1} \tag{3.3}$$

c_0 exceeds 1, and the condition of resonance of the wave with the flow, $u(y) = c$, ceases to hold anywhere. In the limit $J \to 0$ the resonance level y_c approaches y_N from above, $c_0 \approx u_N + J/u'_N$.

When $k > 0$, oscillations are separated into two classes: modes with $c_0 \geqslant 1$ remain neutral, and those with $u_N < c_0 < 1$ become unstable. Indeed, for a not too weak stratification ($k \ll J < J_1(0)$) we obtain: $c = c_0 + c_1 k + c_2 k^2 + \ldots$, where c_0 is determined by equation (3.2),

$$c_1 = \frac{(1-c_0)^2}{2J^2 I_3}, \qquad I_m \stackrel{\text{def}}{=} \int_0^{y_N} \frac{\mathrm{d}y}{(u-c)^m} \quad (m = 3, 4);$$

$$c_2 = \frac{1}{2I_3} \left\{ \frac{(1-c_0)^4}{J^3} \left(1 - \frac{3I_4}{4J {I_3}^2} \right) - \frac{(1-c_0)^3}{J^4 I_3} \right.$$
$$\left. + \frac{1}{J^2} \int_{y_N}^{\infty} \mathrm{d}y_1 (u_1 - c_0)^2 \left[1 - \left(\frac{1-c_0}{u_1 - c_0} \right)^4 \right] + \int_0^{y_N} \mathrm{d}y (u - c_0)^2 \left[\int_0^{y} \frac{\mathrm{d}y_1}{(u_1 - c_0)^2} \right]^2 \right\}.$$

It is easy to see that c_1 is real and negative, and c_2, because of the wave–flow resonance, has a positive imaginary part (it will be recalled that u''_c and I_3 are negative),

$$\operatorname{Im} c_2 = \frac{\pi u''_c (1-c_0)^4}{2 {u'_c}^3 J^2 I_3}, \tag{3.4}$$

i.e. the perturbations are unstable. The subscript c denotes the values of the functions at the resonance point $y = y_c$, where $u(y) = c_r \equiv \operatorname{Re} c$.

In the case of a weak stratification, $J = O(k)$, the difference $|u_N - c| = O(k)$, and the integrals in (3.1), having a resonance denominator, become of $O(k^{-1})$. After corresponding reordering in (3.1) (see equation (A 2)), we obtain, at the main order, the dispersion equation

$$z^2 - \tilde{J} z + \tilde{J}(1-u_N)^2 = 0; \quad z = u'_N (c - u_N)/k, \quad J = k\tilde{J}. \tag{3.5}$$

As with the stronger stratification, the resonance contribution is not involved at the main order. Nevertheless, solutions of (3.5),

$$z = \frac{1}{2} \{ \tilde{J} \pm (\tilde{J}[\tilde{J} - 4(1-u_N)^2])^{1/2} \}, \quad \text{or} \quad c = u_N + \frac{1}{2u'_N} \{ J \pm (J[J - 4k(1-u_N)^2])^{1/2} \}, \tag{3.6}$$

describe unstable oscillation when $0 < J < 4k(1-u_N)^2$. In this region the instability is due to the action of a non-resonance mechanism. When $J \sim 4k(1-u_N)^2$, the contributions to instability from the non-resonance and resonance (the first term on the right-hand side of (A 2)) mechanisms are of the same order, and when $J > 4k(1-u_N)^2$ up to $J = J_1(0) = O(1)$ (see above) the instability is due solely to the wave–flow resonance. For more details about the resonance and non-resonance mechanisms of shear flow instability, see Churilov & Shukhman (2001).

Thus it has been shown that when $k \ll 1$ there is a continuous spectrum of the oscillations whose propagation velocity c_r increases with J from u_N (when $J \to 0$) to infinity (when $J \to \infty$). By the Howard semicircle theorem, only those perturbations which interact with the flow by resonance, i.e. have $c_r < 1$, are unstable and the range of variation of the Richardson parameter $0 < J < J_1(0)$ corresponds to these.

3.2. *The eigen-oscillation spectrum*

Let us show that the structure of eigen-oscillation spectrum found for long-wave perturbations is the same at any k. We begin with neutral oscillations that move faster than the flow. For any real $c \geqslant 1$, in equation (2.2) we have $u''/(u-c) > 0$, and it has real solutions to the left (g_-) and right (g_+) of y_N, each of which satisfies its own boundary condition and has, respectively, a positive and negative logarithmic derivative, so that the second matching condition in (2.3) provides $J = J_c(k) > 0$ for each c and k pair. It is evident that $J_c(k)$ increases monotonically with k, and one can show that it increases monotonically with c too. Therefore, on the (k, J)-diagram the lower boundary of the set of neutral modes with $c \geqslant 1$ is the dispersion curve $J = J_1(k)$ that corresponds to oscillations with $c = 1$.

When $k > 1$, the $J_c(k)$ can be calculated to a good accuracy using the WKB method (compare, for example, curves 1 and 3 in figure 1). On substituting the WKB solutions of equation (2.2),

$$g_+ = K^{-1/2}\exp\left(-\int K\,\mathrm{d}y\right), \quad g_- = K^{-1/2}\sinh\left[\int_0^y K(y_1)\,\mathrm{d}y_1\right]; \quad K^2(y) = k^2 - \frac{u''}{c-u},$$

into the second matching condition in (2.3), we find

$$J_c(k) \approx \frac{2(c-u_N)^2 K(y_N)}{1-\exp\left[-2\displaystyle\int_0^{y_N} K(y)\,\mathrm{d}y\right]} \overset{k \gg 1}{=} 2(c-u_N)^2\,k + O(k^{-1}).$$

It is of interest to mention that in the flow $u(y) = 1 - \mathrm{e}^{-y}$ the WKB method yields the exact result for $J_1(k)$,

$$J_1(k) = \frac{2\kappa \mathrm{e}^{-2y_N}}{1-\mathrm{e}^{-2\kappa y_N}}, \quad \kappa = (1+k^2)^{1/2}.$$

Now let see which oscillations are below the dispersion curve $J = J_1(k)$ on the (k, J)-diagram. Assuming that $u(y) \to 1$ exponentially as $y \to \infty$,

$$u(y) \approx 1 - v\mathrm{e}^{-\alpha y}; \quad v > 0, \quad \alpha > 0,$$

one can find using the perturbation theory (see the first part of Appendix B) that

$$c \approx 1 + \frac{J - J_1(k)}{B_0 J_1(k)} + \frac{\pi \alpha I_0}{B_0 \sin 2\pi q}\left[\frac{\Gamma(q+k/\alpha)}{\Gamma(2q)\Gamma(1+k/\alpha-q)}\right]^2\left[\frac{J-J_1(k)}{v\,B_0 J_1(k)}\right]^{2q},$$

where

$$B_0 = \frac{2}{1-u_N} + I_0\int_0^\infty \frac{\mathrm{d}y u''}{(1-u)^2}{g_0}^2(y) > 0, \quad I_0 = \int_0^{y_N}\frac{\mathrm{d}y}{G^2(y)}, \quad q = \left(1+\frac{k^2}{\alpha^2}\right)^{1/2},$$

and $g_0(y)$ is the eigenfunction of the mode with $c = 1$. It is easy to see that there are neutral modes with $c > 1$ above the curve $J = J_1(k)$ and unstable modes with $c_r < 1$ under it. Indeed, in the last case $[J - J_1(k)]^{2q} = |J_1(k) - J|^{2q}\,\mathrm{e}^{2\mathrm{i}\pi q}$ (see Appendix B), and $c_i \equiv \operatorname{Im} c > 0$.† Hence, the curve $J = J_1(k)$ is *the upper boundary of the instability domain.*

To complete the analysis let us show that there is no other stability boundary (apart form $k=0$, $J=0$, and $J=J_1(k)$). Let there be at a certain $k>0$ a marginally stable mode with $0<c<1$. The matching condition (2.5) may be represented as a dispersion equation

$$(c-u_N)^2 = JF(y_N)G(y_N) \equiv JG^2(y_N)\int_0^{y_N} \frac{\mathrm{d}y}{G^2(y)},$$

which, in particular, shows that the equalities $J=0$ and $c=u_N$ are equivalent: from one follows the other. Assume that $J>0\,(c \neq u_N)$. Then, according to (2.4), $F(0)=0$, and $F'(0)=1/G(0)$, so that $h=G(0)F(y)$ is the solution of the Rayleigh equation with the boundary conditions $h(0)=0$ and $h'(0)=1$, i.e. a real and positive (when $0<y\leqslant y_c$) function, since, by the Rayleigh theorem, when $y>0$ it cannot become zero. For the same reason, $G(y)$ is also positive when $y \geqslant y_c$.

There are two possible variants of the relative position of y_N and y_c. If $y_N<y_c$, then

$$(c-u_N)^2 = J\,\frac{G(y_N)}{G(0)}[G(0)F(y_N)], \quad \arg[(c-u_N)^2] = \arg G(y_N) - \arg G(0), \tag{3.7}$$

and if $y_N>y_c$, then $G(y_N)>0$, and

$$\int_0^{y_N} \frac{\mathrm{d}y}{G^2(y)} = \frac{1}{G(0)}\left[G(0)\int_0^{y_c} \frac{\mathrm{d}y}{G^2(y)}\right] + \int_{y_c}^{y_N} \frac{\mathrm{d}y}{G^2(y)}, \tag{3.8}$$

and the expression in square brackets and the last integral are positive.

With the indentation (from below) of the singular point $y=y_c$, $G(y)$ is continuous, and $G'(y)$ has a jump:

$$G'(y_c+0) - G'(y_c-0) = \frac{\mathrm{i}\pi u_c''}{u_c'}G_c.$$

Upon substituting $G(y)=G_0(y)\exp[\mathrm{i}\varphi(y)]$ when $y<y_c$ and taking into consideration that $G(y)>0$, when $y \geqslant y_c$, from (2.2) we find

$$2\varphi' G_0' + \varphi'' G_0 = 0, \quad \text{or} \quad \varphi' G_0^2 = \text{const} > 0 \quad \text{because} \quad \varphi'(y_c-0) > 0.$$

Thus we have $\varphi<0$ but $\varphi'>0$ when $y<y_c$. On the other hand, $\operatorname{Im} G$ is the solution of equation (2.2), with $\operatorname{Im} G(y_c)=0$, and therefore (by the Rayleigh theorem) it is strictly negative when $0 \leqslant y < y_c$, so that $-\pi<\varphi<0$. Now, using (3.7) and (3.8) it is easy to see that $\operatorname{Im}(c-u_N)^2>0$ when $J>0$, and this is in conflict with the assumption about marginal stability of the mode. Consequently, the lower boundary of the instability domain is $J=0$, and on it $c=u_N$.

Thus we have shown that in the general case the oscillation spectrum is continuous, and it spans the entire first quadrant of the (k, J)-plane. In the Band (see figure 1) the oscillations are unstable, and above it they are neutral. The propagation velocity of perturbations c_r changes with an increase of J from u_N when $J=0$ to 1 when $J \to J_1(k)$ and further monotonically to infinity.

† This reasoning is correct when $2q$ is not equal to an integer. If $2q$ is an integer the formula for c contains $\ln[J-J_1(k)]$ (instead of $[J-J_1(k)]^{2q}$) that does also provide $c_i>0$ as $J<J_1(k)$.

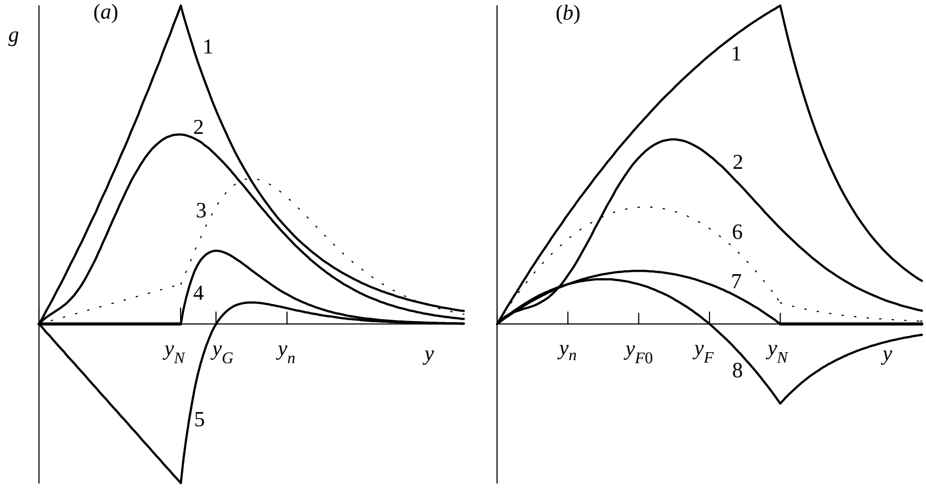

FIGURE 2. Eigenfunctions of neutral oscillations corresponding to the NCP $y = y_n$ when (*a*) $y_N < y_n$, and (*b*) $y_N > y_n$. Curve 1, $k > k_n$; 2, $k = k_n$; 3, $k_G < k < k_n$; 4, $k = k_G$ $(J = \infty)$; 5, $0 < k < k_G$; 6, $k_F < k < k_n$; 7, $k = k_F(J = \infty)$; 8, $0 < k < k_F$. Dots show the eigenfunctions in unstably stratified $(J < 0)$ flows.

4. Stability of flows with NCPs

The velocity profile without inflection points can, however, have points of null curvature in which u'' becomes zero but does not change its sign. In this section we show that, in contrast to flows of a general form where neutral oscillations with $0 < c < 1$ do not exist, in flows with NCPs, to each NCP $y = y_n$ there corresponds a neutral oscillation with the phase velocity $c = c_n = u(y_n) < 1$. Moreover, if the NCP lies above the density jump $(y_n > y_N)$ it creates two little 'islands of stability' in the Band in the form of arcs of the neutral curve $J = J^{(n)}(k)$ that correspond to these oscillations – one arc in the long-wavelength part of the spectrum and the other in the short-wavelength one – and thus reduces the 'area' of the instability domain.

4.1. *Neutral modes with* $0 < c < 1$

It is well known (see, for example, Dikii 1976) that in a homogeneous flow with $u''(y)$ of fixed sign (strictly negative, in particular) there are no eigen-oscillations. If, however, the flow has an NCP, eigen-oscillations do appear (see Appendix C): to each NCP $y = y_n$ there corresponds a neutral oscillation with the phase velocity $c_n = u(y_n)$, specific wavenumber k_n, and eigenfunction $G(y)$. The eigenvalue k_n is found from the condition $G(0) = 0$.

In stratified flows with NCP neutral oscillations of this kind also exist. Indeed, when $c = c_n$, the Rayleigh equation (2.2) has no singularity at the point of phase resonance $u(y) = c$, so that functions $F(y)$ and $G(y)$ are real, and from equation (2.5) there is a real J corresponding to each k, thereby defining the dispersion law $J = J^{(n)}(k)$ for the neutral oscillations associated with NCP $y = y_n$ and propagating with the phase velocity c_n. The sign of $J^{(n)}(k)$ depends on which of F and G has a greater logarithmic derivative at $y = y_N$, i.e. on the behaviour of the functions (that changes with wavenumber k) as well as on the position of the density jump. The behaviour of the eigenfunction $g(y)$ of problem (2.1) in different cases is shown schematically in figure 2.

As mentioned above, when $k = k_n$, $G(0) = 0$ and therefore satisfies both of the boundary conditions in (2.1). Hence, $g_+(y) = G(y)$ as well as $g_-(y) = G(y)$ (each in

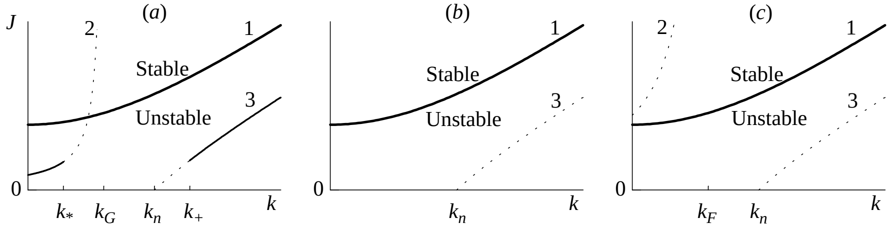

FIGURE 3. Neutral oscillations of flows with an NCP for (*a*) $y_N < y_n$, (*b*) $y_n < y_N < y_{F0}$, (*c*) $y_{F0} < y_N$. Curve 1, the upper boundary of the instability domain, $J = J_1(k)$; 2, the long-wavelength branch of neutral oscillations with $c = c_n$; 3, their short-wavelength branch. The arcs of curves 2 and 3 on which $D(k) > 0$ are plotted as solid lines; the dots correspond to $D(k) < 0$.

its domain of definition), the eigenfunction of (2.1) is smooth (curves 2 in figure 2), and $J^{(n)}(k_n) = 0$. When $k \neq k_n$, however, the eigenfunction is necessarily broken at the density jump (see figure 2). In view of (2.5) and (2.4),

$$J^{(n)}(k) = (u_N - c_n)^2 \left(\frac{F'}{F} - \frac{G'}{G} \right)_{y=y_N} = \frac{(u_N - c_n)^2}{F_N G_N}, \tag{4.1}$$

so that the neutral mode exists under stable stratification ($J > 0$) if $F_N \equiv F(y_N)$ and $G_N \equiv G(y_N)$ have the same sign, and under unstable stratification in the case of opposite signs. The behaviour of $F(y)$ and $G(y)$ as functions of k is analysed in Appendix C: when $k > k_n$ they are both positive everywhere in $0 < y < \infty$, and when $k < k_n$ each of them has exactly one zero in this interval, y_F and y_G respectively, and

$$y_G \leqslant y_n < y_{F0} \leqslant y_F, \tag{4.2}$$

where y_{F0} is a zero of $F(y)$ at $k = 0$. Functions F and G have the same sign outside the interval (4.2) (i.e. when $y < y_G$ or $y > y_F$), and opposite signs inside it.† Hence a change of sign of $J = J^{(n)}(k)$ occurs at $k = k_n$ (when $J^{(n)}(k)$ vanishes) as well as those values of k, if any, at which either y_F or y_G coincides with y_N (and $J^{(n)}(k) \to \infty$).

As applied to stably stratified ($J > 0$) flows, this means that at any position y_N of the density jump there exists a short-wavelength branch of neutral oscillations that is an extension to $J > 0$ of the neutral oscillation with $k = k_n$ of homogeneous flow. At this branch, the eigenfunction of (2.1) is positive (has no nodes) on $0 < y < \infty$ (see figure 2, curves 1 and 2). In accordance with (3.4), there can be also a long-wavelength branch of neutral oscillations with $c = c_n$ that begins at $k = 0$, but only in that case when the integral in (3.2) is positive. The analysis of this condition, in view of (4.2), shows that:

(i) if $y_n < y_N < y_{F0}$, there are no long-wavelength neutral oscillations with $c = c_n$;

(ii) if $y_N < y_n$, the neutral mode exists when $0 \leqslant k < k_G$, where k_G is the value of the wavenumber at which $G_N = 0$;

(iii) finally, if $y_N > y_{F0}$, the neutral mode exists when $0 \leqslant k < k_F$, where k_F is the wavenumber at which $F_N = 0$.

For each of the three cases the branches of the neutral curve are shown in figure 3. Note that on the long-wavelength branch the eigenfunction is certain to have a node (see figure 2, curves 5 and 8).

† Note that the position of points y_n and y_{F0} depends only on the velocity profile $u(y)$, whereas the position of y_F and y_G varies with k.

Thus a neutral mode, having one or two branches in a stably stratified flow, corresponds to each NCP. On the $(k,\ J)$-diagram these branches lie in the first quadrant which is already occupied by the oscillation spectrum found in § 3. Moreover, they fall (at least partially) into the Band (see figure 3) that consists of unstable oscillations. The question arises as to whether this new mode is independent of the 'old' spectrum or is made up of it. To answer this question and to identify the position of the new neutral mode in the complete spectrum of oscillations one should know which oscillations are in its vicinity in the (k, J, c)-space, because every eigen-oscillation is described by these three parameters rather than two.

Using the perturbation theory (for details, see the second part of Appendix B) we find that the imaginary part c_i of the phase velocity has on the neutral curve a second-order zero,

$$c_i = -\frac{A\, u_c^{\mathrm{iv}}}{D(k)}\left[J - J^{(n)}(k)\right]^2, \tag{4.3}$$

where $A > 0$,

$$D(k) = \int_0^\infty \frac{\mathrm{d}y u'' g^2}{(u-c_n)^2} + \frac{2J {g_N}^2}{(c_n - u_N)^3}, \tag{4.4}$$

and $g(y)$ is the eigenfunction of the neutral mode. Thus, the sign of c_i is the same on both sides of the neutral curve. Consequently, the neutral mode with $c = c_n$ *does not belong to the stability boundary.*

As one can see from (4.3), the sign of c_i coincides with the sign of $D(k)$†: only when $D(k) > 0$ are unstable oscillations adjacent to the neutral curve (i.e. at a given k they have close values of J and c_r). If, however, $D(k) < 0$, in the neighbourhood of the neutral mode we obtain $c_i < 0$, which formally corresponds to damped oscillations. It is well known, however, that if the Taylor–Goldstein equation is regarded as a limit (at large Reynolds numbers) of the equations that take into account the dissipation (viscosity and heat conduction), then the limit for dissipative solutions is provided only by unstable and neutral solutions of the Taylor–Goldstein equation (obtained using the indentation of the points of phase resonance by the Landau rule, e.g. Dikii 1976). The solution with $c_i < 0$, however, is not such a limit and will therefore not be considered (as it is 'non-physical'). Accordingly, in figure 3 parts of the neutral curve on which $D(k) > 0$ are plotted as solid lines, and the dots correspond to parts with $D(k) < 0$.

It is easy to verify (see (4.4) and Appendix B) that $D(k) < 0$ everywhere on the neutral curve when $c_n < u_N$, and when $c_n > u_N$, at each of the neutral curve branches, $D(k)$ changes its sign on going from its left to right end: on the long-wavelength branch from plus to minus when $k = k_*$, and on the short-wavelength branch from minus to plus when $k = k_+$ (see figure 3*a*). At the critical points k_* and k_+, where $D(k) = 0$, J and c are related by

$$J - J^{(n)} \approx \frac{1}{2}\frac{\partial^2 J}{\partial c^2}(c - c_n)^2, \quad \text{or} \quad c - c_n \sim \left(J - J^{(n)}\right)^{1/2} \tag{4.5}$$

(see (B 11)). Figure 4 presents the trajectories (with a change of J) of all solutions (4.5) on the complex plane c, but, as has already been pointed out, only the solutions with $c_i \geqslant 0$ (shown by solid lines) have physical meaning.

† Since $u''(y) \leqslant 0$, in the NCP the first (after u'') non-zero derivative has an even order and is negative.

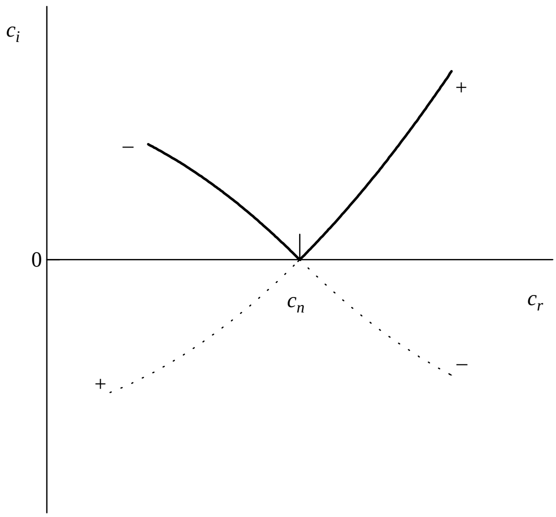

FIGURE 4. Trajectories of $c(J)$ near the neutral curve when $D(k)=0$. Signs near the lines designate the sign of difference $(J-J^{(n)}(k))$.

Now we have to make clear the position filled by the NCP-associated neutral mode in the oscillation spectrum. Let start with its short-wavelength end ($k \gg 1$), and then analyse the whole picture of the flow stability.

4.2. *Short-wavelength perturbations*

For the short-wavelength region the problem of the spectrum of flow eigen-oscillations can be solved analytically. When $k \gg 1$ the term with u'' in equation (2.2) may be regarded as a perturbation, and the solution can be constructed in the form of an expansion in terms of k^{-1}. We obtain (up to normalization)

$$g_+ = \mathrm{e}^{-ky}\left\{1 + \frac{1}{2k}\int_y^\infty \frac{\mathrm{d}y_1 u_1''}{u_1 - c}\left[1 - \mathrm{e}^{-2k(y_1-y)}\right] + O(k^{-2})\right\},$$

$$g_- = \mathrm{e}^{ky}\left\{1 - \mathrm{e}^{-2ky} + \frac{1}{2k}\int_0^y \frac{\mathrm{d}y_1 u_1''}{u_1 - c}\left[1 - \mathrm{e}^{-2k(y-y_1)}\right](1 - \mathrm{e}^{-2ky_1}) + O(k^{-2})\right\}.$$

Using the second matching condition (2.3) and omitting exponentially small terms we arrive at the dispersion equation

$$(c-u_N)^2 = \frac{J}{2k}\left\{1 - \frac{1}{2k}\int_0^{y_N} \frac{\mathrm{d}y u''}{u-c}(1-\mathrm{e}^{-2ky})\mathrm{e}^{-2k(y_N-y)} - \frac{1}{2k}\int_{y_N}^\infty \frac{\mathrm{d}y u''}{u-c}\mathrm{e}^{-2k(y-y_N)}\right\} + \cdots$$

By solving it using the method of successive approximations ($c=c^{(0)}+c^{(1)}+\cdots$) we find

$$\left(c^{(0)} - u_N\right)^2 = \frac{J}{2k},$$

$$c^{(1)} = -\frac{c^{(0)}-u_N}{4k}\left[\int_0^{y_N} \frac{\mathrm{d}y u''}{u-c^{(0)}}(1-\mathrm{e}^{-2ky})\mathrm{e}^{-2k(y_N-y)} + ⨍_{y_N}^\infty \frac{\mathrm{d}y u''}{u-c^{(0)}}\mathrm{e}^{-2k(y-y_N)} + \frac{\mathrm{i}\pi u_c''}{u_c'}\mathrm{e}^{-2k(y_c-y_N)}\right], \quad (4.6)$$

where $⨍$ denotes a Cauchy principal value integral.

Thus the instability at $k \gg 1$ is caused by the wave–flow phase resonance, and since $u'' \leqslant 0$, the imaginary part of the phase velocity,

$$c_i \approx -\frac{\pi}{4k}\frac{u_c''}{u_c'}\left(c^{(0)} - u_N\right) \mathrm{e}^{-2k(y_c - y_N)} \equiv -\frac{\pi}{2}\frac{u_c''}{u_c'}\frac{J^{1/2}}{(2k)^{3/2}} \mathrm{e}^{-2k(y_c - y_N)}, \tag{4.7}$$

takes positive values only when $u_N < c^{(0)} < 1$, i.e. in the Band as in flows without the NCP. The growth rate $\gamma = c_i k$ becomes zero at the edges of the Band, when $J = 0$ (i.e. $c^{(0)} = u_N$), as well as when $c^{(0)} \to 1$ (i.e. $y_c \to \infty$). If an NCP-associated neutral mode has velocity in this range (i.e. if $u_N < c_n < 1$), the growth rate also vanishes at the neutral curve $J = J^{(n)}(k)$, remaining positive on both sides of it (as also follows from (4.3), since $D(k) > 0$ when $k \gg 1$). This is the key difference in the structure of the instability domain for flows with and without the NCP (as well as with the NCP lying below the density jump, at $y_n < y_N$). Such a neutral mode is the inner limiting point of a set of unstable modes (rather than the stability boundary). The corresponding neutral curve dissects the Band into stripes, in each of which the growth rate changes from zero at the edges to some maximum (for a given k) value. Let us determine these maxima.

Differentiating (4.7) with respect to J, we obtain the extremum condition:

$$\frac{\partial \gamma}{\partial J} = -\frac{\pi}{4}\frac{\partial c^{(0)}}{\partial J}\left[\frac{u_c''}{u_c'} + \left(c^{(0)} - u_N\right)\left(\frac{u_c'''}{u_c'^2} - \frac{u_c''^2}{u_c'^3} - 2k\frac{u_c''}{u_c'^2}\right)\right] \mathrm{e}^{-2k(y_c - y_N)} = 0,$$

from which it is evident that the minimum (zero) value of the growth rate is indeed attained on the neutral curve, at $c^{(0)} = c_n$, when $u_c'' = 0$, and $u_c''' = 0$. Maximum values (γ_m), however, are reached near the lower ($J = 0$) boundary of the instability domain,

$$\gamma_{m1}(k) = -\frac{\pi u_N''}{8\,\mathrm{e}k} + O(k^{-2}) \quad \text{at} \quad c^{(0)} = u_N + \frac{u_N'}{2k} + O(k^{-2}) \quad \text{and} \quad J = \frac{u_N'^2}{2k} + O(k^{-2}),$$

and immediately above the neutral curve (assuming that $y_n - y_N = O(1)$),

$$\gamma_{m2}(k) = -\frac{\pi u_n^{\mathrm{iv}}}{8\,\mathrm{e}^2 k^2 u_n'}(c_n - u_N)\mathrm{e}^{-2k(y_n - y_N)} \quad \text{at} \quad c^{(0)} = c_n + \frac{u_n'}{k} + O(k^{-2}).$$

With an increase of k, γ_{m1} decreases as $1/k$, and γ_{m2} decreases exponentially.

4.3. *Oscillation spectrum of flows with an NCP*

Now we analyse the flow oscillation spectrum as a whole. Proceeding in the same manner as in § 3.2, it is easy to verify that in flows with an NCP the continuous spectrum $J = J_c(k)$ of the neutral oscillations with $c \geqslant 1$ remains, and that the lower curve of this family, $J = J_1(k)$, again serves as the upper boundary of the instability domain, i.e. unstable modes are adjacent to it from below along its entire length. Further, in flows with an NCP, there are also no marginally stable modes with $J > 0$ and $c < 1$. These results, together with the results of an analytical investigation of the long-wavelength (§ 3.1) and short-wavelength (§ 4.2) asymptotics, lead us to conclude that in flows with an NCP the instability domain on the (k, J)-plane also lies in the Band. The velocity c_r of unstable oscillations also changes with J from u_N when $J = 0$ to 1 when $J \to J_1(k) - 0$. The distinctive feature of flows with an NCP is that each NCP has its corresponding neutral mode of oscillations with phase velocity $c = c_n$ and with the dispersion law $J = J^{(n)}(k)$. Let us find the position of this mode in the picture of eigen-oscillations of the flow.

First, we consider the case where there is a single NCP lying above the density jump ($c_n > u_N$). It is convenient to use a specific model flow for illustrating the relevant

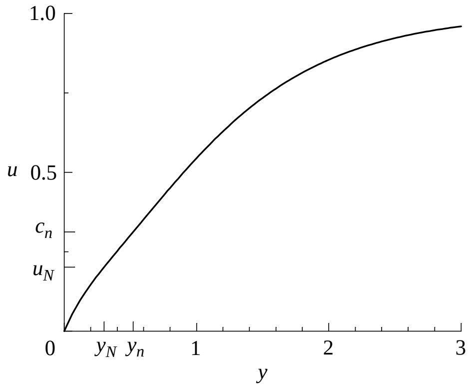

FIGURE 5. The velocity profile of the model flow (4.8) with $a = 0.5$.

reasoning and conclusions. For this purpose, of the one-parameter family of flows with an NCP,

$$u(y) = 1 - \frac{2z^2 - 9az + 18a^2}{2 - 9a + 18a^2} z; \quad z = \mathrm{e}^{-\mu y}, \quad \mu = \frac{2 - 9a + 18a^2}{6(1 - 3a + 3a^2)}, \tag{4.8}$$

a flow with $a = 0.5$ and a density jump at $y_N = 0.3$ is chosen (see figure 5). The flow parameters are

$$\mu = \tfrac{4}{3}; \quad u_N = 0.201576, \quad y_n = 0.519860, \quad c_n = \tfrac{5}{16} = 0.3125;$$
$$k_* = 0.062718, \quad k_G = 0.110901, \quad k_n = 0.212870, \quad k_+ = 0.293323.$$

Results of calculations for this model are presented in figures 6 and 7.

At the long-wavelength ($k \ll 1$) end of the spectrum, as is evident from (3.2) and (3.4), one mode corresponds to each $J > 0$, and c_r increases monotonically with J from u_N to ∞, and c_i is zero above the Band, is non-negative in the Band and changes continuously but not monotonically: it tends to zero at the edges of the Band, as well as having a second-order zero within the Band, when $c = c_n$, i.e. on the long-wavelength branch of the neutral curve $J = J^{(n)}(k)$. The analysis in § 4.1 showed that when $D(k) > 0$, near the neutral curve there are only unstable modes; therefore, we conclude from continuity that $c_i(J)$ has on the neutral curve a second-order zero for each k from the range $0 \leqslant k < k_*$.† When $k = k_*$, c_i also becomes zero, but now according to (4.5). At the short-wavelength end of the spectrum the picture is similar (see § 4.2), in particular, $c_i(J)$ exactly has a second-order zero on the (short-wavelength branch of) neutral curve for each k from the interval $k_+ < k < \infty$ (figure 7c, d).

Thus, when $0 \leqslant k < k_*$ or $k_+ < k < \infty$ the Band includes the neutral mode that has a phase velocity c_n and is (for each k from the intervals indicated) the inner limiting point of a set of unstable modes (see figure 6). It can be said that the NCP modifies the instability domain: it now occupies not the entire Band but a band that is cut from the left and right by the corresponding arcs of the neutral curve $J = J^{(n)}(k)$ (‘solid’ arcs of lines 2 and 3 in figures 3(a) and 6). The points $(k_*, J^{(n)}(k_*))$ and $(k_+, J^{(n)}(k_+))$ on the (k, J)-diagram are bifurcation points in which the neutral mode ‘separates’ from

† A numerical calculation of the model (4.8) lends support to this; however, its results for $0 < k < k_*$ are extremely inconvenient for a graphical representation; therefore, the plots are given for $k > k_*$ only.

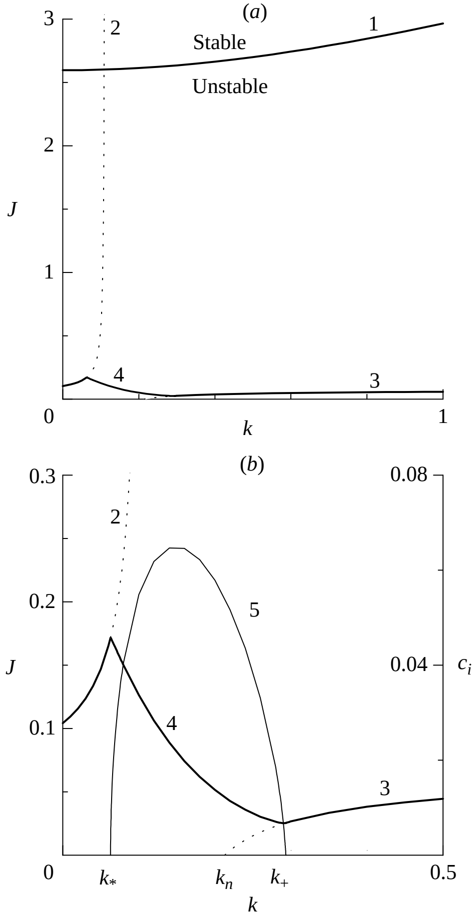

FIGURE 6. Instability domain of the flow (4.8): (*a*) the (k, J)-diagram; (*b*) lower part of the diagram (enlarged). Curve 1, the upper boundary of the instability domain $J = J_1(k)$; 2, the long-wavelength branch of neutral oscillations; 3, their short-wavelength branch; 4, oscillations with $c_r = c_n$ and $k_* < k < k_+$; 5, dependence of c_i on k for these oscillations. Dots show the arcs of the curves 2 and 3 on which $D(k) < 0$.

the continuous spectrum (in the sense that it fails to have neighbouring oscillations in the (k, J, c)-space), and the unstable mode with $c_r = c_n$ appears (see figure 6*b*).

To check this, take k from the interval $k_* < k < k_+$ and look at the behaviour of the solution of the problem (2.1) with a change of J from $J_1(k)$ to 0. In the space $(k,\ J,\ c)$ we obtain a certain trajectory, along which c_r changes continuously from 1 to u_N taking, among others, the value c_n as well. At the ends of this trajectory $c_i \to 0$, and at the inner points c_i is positive. Indeed, our trajectory does not intersect the neutral curve (even if it exists for a given k), since $D(k) < 0$, and in the neighbourhood of the neutral curve there are no unstable modes. There are no other neutral modes, however, as shown in §3.2, so that at the inner point of the trajectory $c_i(J)$ cannot become zero. If k is close to k_* or k_+, relation $c_i(J)$ has a characteristic two-hump form (figure 7*b*), but c_i does not become zero between the humps. If, however, k is not close to k_* and k_+ there is only one maximum (figure 7*a*). Consequently, in those spectral regions where $D(k) < 0$ (i.e. when $k_* < k < k_G$ or $k_n \leqslant k < k_+$), the neutral

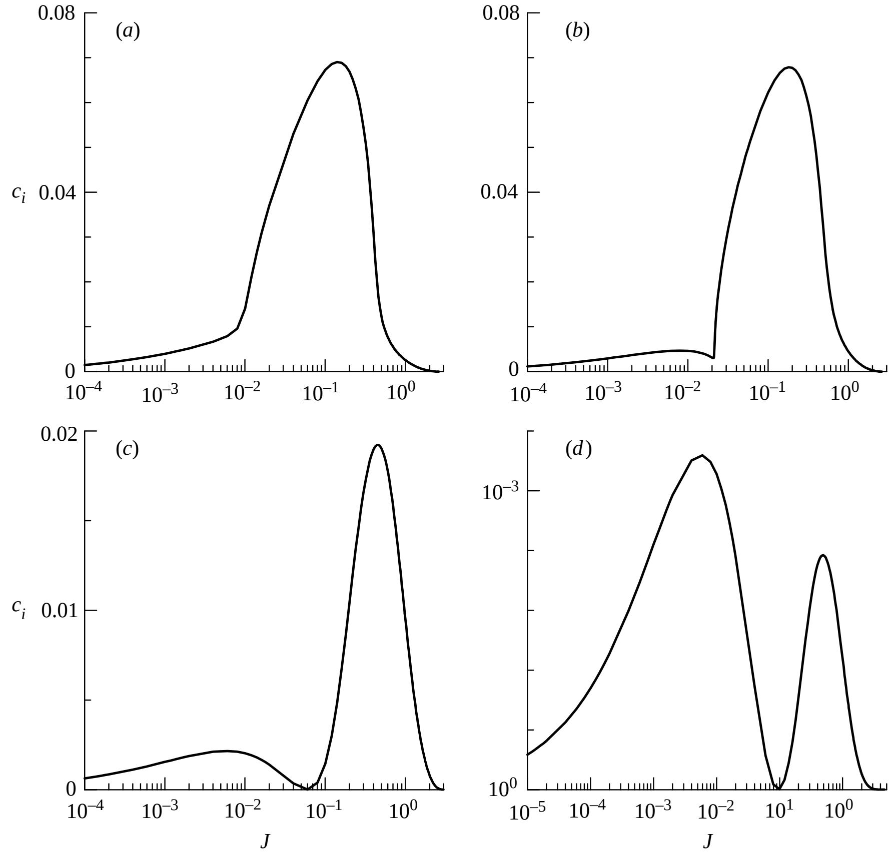

FIGURE 7. Dependence of c_i on J for different k: (a) $k=0.2$; (b) $k=0.27$; (c) $k=1.0$; (d) $k=3.0$.

mode is isolated and supplements, rather than modifying, the previously obtained oscillation spectrum.

If, however, the NCP is such that $c_n < u_N$, then on the entire neutral curve we have $D(k) < 0$ (see (4.4)), and similar reasoning shows that in this case the oscillation spectrum, described in § 3, is not modified but is only supplemented by the isolated neutral mode with $c = c_n$.†

The above analysis is readily generalized to flows with several NCPs. Each NCP with $c_n > u_N$ leaves a pair of cuts in the Band on which c_i becomes zero and, possibly, supplements the spectrum by an isolated neutral mode. As long as the number of NCPs with $c_n > u_N$ is finite, these cuts simply dissect the long- and short-wavelength ends of the Band into stripes and thereby 'modulate' the $c_i(J)$ relation without substantially altering the area of instability domain. If, however, on the velocity profile there are continuous arcs composed of NCPs with $c_n > u_N$, then the corresponding cuts merge together into continuous regions, thus substantially decreasing the part of the Band which is occupied by the instability domain. The way in which this is taking place will be shown in the next section by considering the flow (1.2).

† Note that this mode is merely a formal solution of (2.1). The question of the physical meaning of the neutral solutions with $D(k) < 0$ and, in particular, as to how they match the solution of the dissipative equations, demands special investigation and still remains open.

5. Stability of a flow with a piecewise-linear velocity profile

In the flow (1.2) we have $u''=-\delta(y-1)$ so that the velocity profile is made up entirely of NCPs and a single point ($y=1$) of infinite curvature. Equation (2.2) becomes an equation $\mathrm{d}^2g/\mathrm{d}y^2-k^2g=0$ plus matching conditions for $y=1$:

$$g(1-0)=g(1+0), \quad g'(1+0)-g'(1-0)+\frac{g(1)}{1-c}=0, \tag{5.1}$$

The solution of the problem (2.2), (2.3) has the form

$$\left.\begin{array}{l} g=A\sinh ky \quad \text{to the left of the matching points } y=y_N \text{ and } y=1,\\ g=B_1\,\mathrm{e}^{ky}+B_2\,\mathrm{e}^{-ky} \quad \text{between matching points,}\\ g=\mathrm{e}^{-ky} \quad \text{to the right of the matching points,} \end{array}\right\} \tag{5.2}$$

where A, B_1 and B_2 are arbitrary (for the time being) coefficients.

There are two variants of relative position of the matching points. If the density jump is above the layer of the velocity shear ($y_N \geqslant 1$), then matching of the solutions (5.2) leads to a dispersion equation

$$\left[1-\frac{J(1-E)}{2k(1-c)^2}\right]\left[1-\frac{t}{k(1-c)}\right]+\left[1-\frac{J(1+E)}{2k(1-c)^2}\right]t=0, \tag{5.3}$$

where $t=\tanh k$, and $E=\exp[-2k(y_N-1)]$. This cubic-in-c equation has only real roots (see Appendix D), which suggests the presence of neutral oscillations only.

If, however, $y_N<1$, the matching procedure leads to the relations

$$B_1=\frac{\mathrm{e}^{-2k}}{2k(1-c)}, \quad B_2=1-\frac{1}{2k(1-c)} \tag{5.4}$$

and to the dispersion equation

$$\left[1-\frac{1-E}{2k(1-c)}\right]\left[1-\frac{Jt}{k(u_N-c)^2}\right]+\left[1-\frac{1+E}{2k(1-c)}\right]t=0, \tag{5.5}$$

where $t=\tanh(ky_N)$, and $E=\exp[-2k(1-y_N)]$. It is a cubic (for c) equation that has a pair of complex conjugate roots (one of them corresponding to the unstable mode), if (see Appendix D)

$$\max(J_-,0) < J < J_+,$$

where

$$\left.\begin{array}{l} J_\pm=\dfrac{1+t}{4kt}\left\{(d+E)^2+\dfrac{5Et}{1+t}(d+E)-\dfrac{E^2t^2}{2(1+t)^2}\pm 4\left(\dfrac{Et}{1+t}\right)^{1/2}\left[d+E+\dfrac{Et}{4(1+t)}\right]^{3/2}\right\},\\ d=2k(1-u_N)-1. \end{array}\right\} \tag{5.6}$$

With increasing k, the functions $J_\pm(k)$ increase monotonically; $J_+\geqslant 0$ and J_- becomes positive when $k>k_0$, where k_0 is the solution of the equation

$$\mathrm{e}^{-2k_0}=1-2k_0(1-u_N). \tag{5.7}$$

We now also give the asymptotic formulae for the case where $1-u_N=O(1)$. When $k\ll 1$, for the unstable mode we have

$$\left.\begin{array}{l} c=u_N+\dfrac{J}{2}+\dfrac{\mathrm{i}}{2}[J(J_+-J)]^{1/2},\\ J_+\approx 4(1-u_N)^2k\left[1-(1-u_N)\left(\dfrac{11}{3}-\dfrac{1}{u_N}\right)k\right]. \end{array}\right\} \tag{5.8}$$

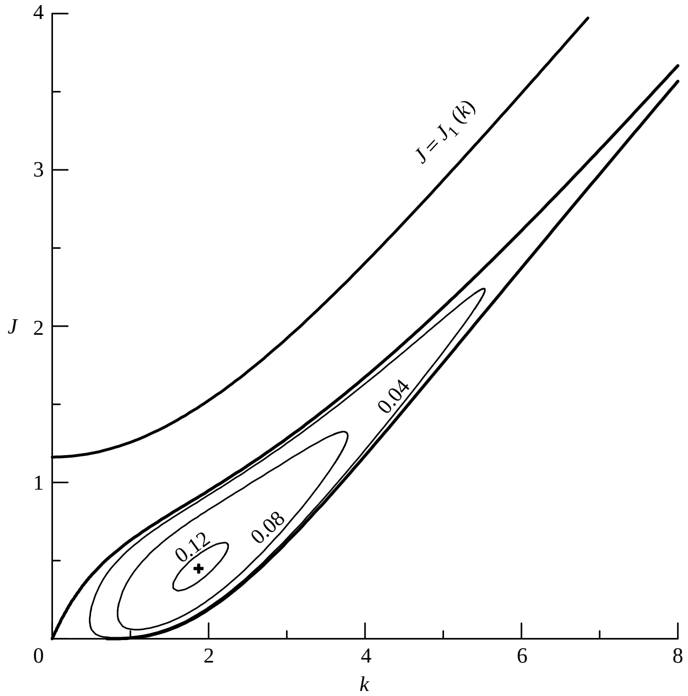

FIGURE 8. Instability domain of the flow with the piecewise linear velocity profile (1.2). Thin lines show contours of constant growth rate. The cross indicates the point of maximum growth rate ($\gamma = 0.1233$ at $k = 1.87$ and $J = 0.448$).

When $k \gg 1$,

$$J_{\pm} \approx 2k\left(1 - u_N - \frac{1}{2k}\right)^2 \pm 4k^{1/2}(1 - u_N)^{3/2}\,\mathrm{e}^{-k(1-u_N)}, \tag{5.9}$$

i.e. the boundaries of the instability domain approach each other exponentially (and the growth rate is, accordingly, exponentially small). The instability domain of the flow (1.2) for $u_N = \tanh\,(1/2)$ is shown in figure 8 (and is shaded in figure 1).

Equation (5.5) describes all flow oscillations, including neutral ones. On rewriting it as

$$\frac{Jt}{k} = (c - u_N)^2\left[1 + t + \frac{2Et}{2k(c-1) + 1 - E}\right] = (c - u_N)^2\left[1 + t + \frac{2Et}{2k(c - u_N) - d - E}\right], \tag{5.10}$$

it is easy to verify that for each k there exists a continuous spectrum $J = J_c(k)$ of the neutral oscillations with $c \geqslant 1$ that occupies the upper part of the first quadrant of the (k, J)-plane. Between the lower ($c = 1$) boundary of this spectrum (see figure 8),

$$J = J_1(k) = (1 - u_N)^2 k\left(\frac{1}{t} + \frac{1+E}{1-E}\right) = (1 - u_N)^2 k[\coth ky_N + \coth k(1 - y_N)],$$

and the abscissa axis ($J = 0$) lies the Band that includes the instability domain. Comparison with (5.6) and (5.9) shows that $J_1(k)$ is separated from the actual upper boundary $J = J_+(k)$ of the instability domain for all k by a gap of width $O(1)$ which, in the light of the results from the preceding section, must be occupied by cuts that are left on the Band by long-wavelength branches of neutral curves corresponding to neutral oscillations with $u_N < c_n < 1$. In a similar manner, the gap between the abscissa axis and the actual lower boundary of the instability domain ($J = J_{-}(k)$ when $k > k_0$) must be occupied by cuts from short-wavelength branches of the same neutral curves. Let us consider the neutral curves corresponding to these oscillations in an attempt to demonstrate that this is indeed the case.

We start from the short-wavelength branches. In Appendix C it was established that in a homogeneous ($J = 0$) flow, one neutral oscillation corresponds to each NCP. In the flow (1.2), the velocity profile consists almost entirely of NCPs; therefore, here to each value of c_n from the interval $0 \leqslant c_n < 1$ a neutral oscillation with the wavenumber $k_n(c_n)$ corresponds, that is inferred from the equation (cf. (5.7))

$$\mathrm{e}^{-2k_n} = 1 - 2k_n(1 - c_n)$$

and increases monotonically from zero (when $c_n = 0$) to k_0 (when $c_n = u_N$), and further to infinity (when $c_n \to 1$). In a stratified ($J > 0$) medium, each oscillation of this sort gives rise to a short-wavelength branch of the neutral oscillations propagating with the same velocity. To put it another way, a neutral curve issues from each point of the positive abscissa semiaxis: when $0 < k < k_0$, neutral curves with $c_n < u_N$ start (that do not modify the stability domain), and when $k > k_0$, neutral curves with $c_n > u_N$ start, which do provide the cuts at the right-hand end of the Band.

Unlike the short-wavelength branch, the long-wavelength branch does not exist for any relation between c_n and u_N (see figure 3) but only when either $c_n > u_N$ or $y_N > y_{F0} = 1/c_n > 1$. The latter case is of no interest, as there is no instability at such a position of the density jump. For the modes with $u_N < c_n < 1$, however, in the limit $k \to 0$ we obtain from (5.10) the spectrum $J = c_n(c_n - u_N)/u_N$ which, of course, coincides with (3.2). These are neutral oscillations, and since in the flow (1.2) $u'' = 0$ almost everywhere, in some neighbourhood (in k) of the positive J-semiaxis oscillations are also neutral. Indeed, from each point of this semiaxis in the interval $0 < J < (1 - u_N)/u_N \equiv J_1(0)$ a branch of neutral oscillations with the value of c_n corresponding to this J starts. These branches merge to constitute the region of neutral oscillations adjacent to the J-axis.

With $c = c_n$ fixed in (5.10), it is possible to study how $J^{(n)}(k)$ increases along the neutral curve, tending to infinity when $k \to k_G$, where k_G is determined from the equation

$$2k_G(c_n - 1) + 1 - E = 0, \quad \text{or} \quad \mathrm{e}^{-2k_G(1-y_N)} = 1 - 2k_G(1 - c_n),$$

and, as is readily verified, can vary from zero (when $c_n \to u_N$) to infinity (when $c_n \to 1$) remaining less than $k_n(c_n)$ everywhere.

Thus the NCPs of the flow (1.2) generate two families of curves on the (k, J)-plane which are made up of the long- and short-wavelength branches of the neutral curves, respectively. Let us demonstrate that the envelopes of these families do represent the boundaries $J = J_{\pm}(k)$ of the actual instability domain. Differentiating (5.10) with respect to c we obtain

$$\left.\begin{aligned} &(c - u_N)^2 - (c - u_N)\left(2d + \frac{2 + t}{1 + t}E\right) + (d + E)\left[d + E - \frac{2Et}{1 + t}\right] = 0 \\ &\text{or} \\ &c = u_N + d + E - \frac{Et}{2(1 + t)} \pm \left(\frac{Et}{1 + t}\left[d + E - \frac{Et}{4(1 + t)}\right]\right)^{1/2}. \end{aligned}\right\} \quad (5.11)$$

On excluding c from (5.10) and (5.11), we find the equation for the envelopes which, as can be verified, exactly coincide with (5.6). Simultaneously, we have obtained formula (5.11), showing how the phase velocity varies along the boundaries of the instability domain.

Finally, from (4.4) using (5.2) and (5.4) we find

$$D(k) = -\frac{g^2(1)}{(1-c)^2} + \frac{2J{g_N}^2}{(c-u_N)^3}$$
$$= -\frac{\mathrm{e}^{-2k}}{(1-c)^2} + \frac{2J}{(c-u_N)^3}\left(\frac{\mathrm{e}^{-2k+ky_N}}{2k(1-c)} + \left[1 - \frac{1}{2k(1-c)}\right]\mathrm{e}^{-ky_N}\right)^2.$$

On setting D equal to zero and transforming the resulting equation using (5.10), we arrive at (5.11). Consequently, the neutral curve arcs on which $D > 0$ (cuts) end just on the boundaries (5.6) of the instability domain, so that the gaps between $J_1(k)$ and $J_+(k)$ in the upper part of the Band, and between $J_-(k)$ and $J = 0$ in the lower part are occupied just by NCP-related cuts.

6. Discussion

The main result of this study is that, with minimal assumptions about the velocity profile ($u' > 0, u'' \leqslant 0$), it appears possible to obtain sufficiently detailed information about flow stability. In flows of a general form ($u'' < 0$ everywhere), the instability domain on the (k, J)-plane occupies a band $0 < J < J_1(k)$, having width of the order of unity at the long-wavelength end and expanding infinitely when $k \to \infty$ (figure 1). The velocity c_r of unstable modes varies from u_N when $J \to 0$ to 1 when $J \to J_1(k)$. The instability growth rate tends to zero at the edges of the Band and is positive everywhere inside it.

The class of flows considered in this study is stable in a homogeneous medium, so that instability is governed by stratification: weak stratification generates it, at all scales simultaneously, from $k = 0$ to $k = \infty$, and strong stratification suppresses it but only gradually, starting from long-wavelength perturbations. The fact that the stable stratification can destabilize a flow that is stable in a homogeneous medium was discovered by Thorpe (1969) and was discussed by a number of authors (see, for example, Makov & Stepanyants 1984, 1985; Kozyrev & Stepanyants 1991; Caulfield 1994). Its physical interpretation is usually based on waves of negative energy (e.g. Kozyrev & Stepanyants 1991; Caulfield 1994): such waves (which are associated with velocity shear and are stable in themselves) do interact linearly with internal waves (that are due to buoyancy and have positive energy), and this interaction yields instability. Regarding stabilization, its mechanism is simple and clear. As is evident from the dispersion equations (3.2) and (4.6) corresponding to the limiting cases of long and short waves, with an enhancement of stratification, the velocity c_r of the perturbation (with a given k) increases. Its resonance level $y = y_c$ is thus expelled increasingly closer to the flow periphery, and when stratification becomes so strong that $J \geqslant J_1(k)$, the wave is no longer in resonance with the flow and turns into a neutral oscillation of the medium.

This instability pattern can change in some details if on the flow velocity profile there is a point (or points) of zero curvature $y = y_n$, where $u'' = 0$. We have shown that to each NCP there corresponds a neutrally stable solution of the problem (2.1) which describes the neutral oscillation mode, having a phase velocity $c_n = u(y_n)$ and represented on the (k, J)-plane by the neutral curve $J = J^{(n)}(k)$ (see figure 3). If the

NCP lies below the density jump ($c_n < u_N$), the neutral mode is isolated (i.e. there are no other eigen-oscillations in its neighbourhood in the (k, J, c)-space) and does not influence the configuration of the instability domain. If, however, $c_n > u_N$, the neutral curve has two branches (figure 3*a*), the arcs of which when $0 \leqslant k < k_*$ and $k_+ < k < \infty$ (i.e. where $D(k) > 0$) belong to the Band in the sense that for each k from the intervals indicated the neutral mode is an inner limiting point of a set of unstable modes.

As a result, each NCP lying above the density jump modifies the instability domain, taking out of the Band one neutral 'cut' at the left (long-wavelength) and right (short-wavelength) ends. If the number of such NCPs is finite, then the ends of the Band are merely cut into a corresponding number of stripes (see figures 3(*a*) and 6), on the boundaries between which the growth rate becomes zero. If, however, as in the flow (1.2), whole segments of the velocity profile consist of NCPs, then the cuts corresponding to these NCPs produce continuous regions inside the Band which narrow greatly the instability domain (figure 8).

The NCP-related changes in the configuration of the instability domain have a simple physical interpretation if it is taken into consideration that shear flow instability is the result of a combined action of two mechanisms: the resonance and non-resonance mechanisms (see, for example, Churilov & Shukhman 2001). First we consider the resonance mechanism as the better studied one. It is based on the interaction of the perturbation wave with a so-called critical layer, a narrow neighbourhood of the resonance level $y = y_c$, on which the flow velocity coincides with the wave velocity, $u(y_c) = c_r$. The result of the resonance interaction (the absorption or enhancement of the wave by the flow in the critical layer) depends on which number of 'fluid particles' is larger: those which keep slightly ahead of the wave or those which lag slightly behind. Therefore, it is highly sensitive to the structural details of the flow. In terms of the plasma–hydrodynamic analogy (Andronov & Fabrikant 1979), the indicator is provided by the sign of the derivative of the 'distribution function' $f(u) = |u'|^{-1}$ of fluid particle velocities: when $\mathrm{d}f/\mathrm{d}u \equiv -u''/u'^3 > 0$ the wave is enhanced by resonance, and when $\mathrm{d}f/\mathrm{d}u < 0$ it is absorbed by the flow.

In the class of flows considered above $u'' \leqslant 0$; therefore, there is no resonance absorption, and the 'non-resonance' instability domain is supplemented and expanded by the region of resonance enhancement of the oscillations. In the vicinity of each NCP the number of leading and lagging fluid particles is exactly equal ($u'' = 0$); therefore, the waves having a corresponding velocity, $c_r = c_n$, are not enhanced by resonance and are unstable only in that range of k where instability is ensured by the non-resonance mechanism. In the limiting case of the velocity profile (1.2) that is composed of the NCPs, none of the waves is enhanced by resonance, so that figure 8 presents the 'non-resonance' instability domain (it is also shaded in figure 1), while its complement to the Band is, in essence, the region of potential (but not realized in the flow (1.2)) resonance enhancement.

It should be noted that of importance for the non-resonance mechanism is the existence of a velocity shear, whereas it is almost insensitive to the structural details of the flow. This is indicated by all of the cases where it is possible to separate the resonance and non-resonance contributions to instability (see, for example, Churilov & Shukhman 2001; Drazin & Howard 1962). One can also verify that this is the case by comparing formulae (5.8) describing the instability of the flow (1.2) for $k \ll 1$, with the non-resonance contribution (3.6) to the instability of an arbitrary flow – they are in fact identical. This suggests that figure 8 also reflects (at least qualitatively) the configuration of the non-resonance part of the instability domain of flows of a general form without inflection points.

In closing we make a brief remark regarding the density profile. The two-layer model (1.1) should, obviously, be regarded as a limiting case where the ratio of the scale l of density variation to the scale L of velocity variation tends to zero. The chief merit of this model is that it makes it possible to separate in space two such exceptionally important (for the perturbation dynamics) factors as the wave-flow resonance and the stratification of the medium. One can expect that eigen-oscillations found above will also remain (with some modifications) in a flow with a finite $l \ll L$. The most modified of them will be those oscillations whose resonance level y_c is embedded in essentially inhomogeneous layers, $|y_c - y_N| = O(l)$, as well as oscillations with $kl > 1$. As the result, the lower boundary of the instability domain will rise from $J = 0$ to $J = O(kl)$, hence there would appear an instability threshold in J, and for each J the spectrum of unstable perturbations would become bounded in k from above (and, possibly, from below). It is highly plausible that new kinds of oscillations (unstable ones among them) would appear which have no with counterpart in the two-layer model. What oscillations would appear and what further changes would be experienced by the instability domain for a finite (but small) l could be revealed only by a special investigation.

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Appendix A. Solving the problem (2.6) for $k \ll 1$

We seek the solution at the right (+) and left (−) of the point $y = y_N$ in the form of an expansion in terms of k:

$$f_\pm = f_\pm^{(0)} + k f_\pm^{(1)} + k^2 f_\pm^{(2)} + \dots;$$

$$f_+^{(0)}(\infty) = 1, \quad f_+^{(i)}(\infty) = 0, \quad f_-^{(0)}(0) = f_-^{(i)}(0) = 0, \quad i = 1, 2, \dots$$

At $O(1)$

$$\frac{\mathrm{d}}{\mathrm{d}y}\left[(u-c)^2 \frac{\mathrm{d}f_\pm^{(0)}}{\mathrm{d}y}\right] = 0;$$

$$f_+^{(0)}(y) = 1, \quad (u-c)^2 \frac{\mathrm{d}f_-^{(0)}}{\mathrm{d}y} = B = \text{const}, \quad f_-^{(0)} = B\int_0^y \frac{\mathrm{d}y_1}{(u_1-c)^2}.$$

At $O(k)$ at the right of y_N we have

$$\frac{\mathrm{d}}{\mathrm{d}y}\left[(u-c)^2 \frac{\mathrm{d}f_+^{(1)}}{\mathrm{d}y}\right] = 2(u-c)\frac{\mathrm{d}}{\mathrm{d}y}\left[(u-c)f_+^{(0)}\right] = \frac{\mathrm{d}}{\mathrm{d}y}[(u-c)^2 - (1-c)^2],$$

$$(u-c)^2 \frac{\mathrm{d}f_+^{(1)}}{\mathrm{d}y} = (u-c)^2 - (1-c)^2, \quad f_+^{(1)} = -\int_y^\infty \mathrm{d}y_1\left[1 - \left(\frac{1-c}{u_1-c}\right)^2\right],$$

and at the left

$$\frac{\mathrm{d}}{\mathrm{d}y}\left[(u-c)^2 \frac{\mathrm{d}f_-^{(1)}}{\mathrm{d}y}\right] = 2(u-c)\frac{\mathrm{d}}{\mathrm{d}y}\left[(u-c)f_-^{(0)}\right] = \frac{\mathrm{d}}{\mathrm{d}y}\left[(u-c)^2 f_-^{(0)}\right] + (u-c)^2\frac{\mathrm{d}f_-^{(0)}}{\mathrm{d}y},$$

$$(u-c)^2 \frac{\mathrm{d}f_-^{(1)}}{\mathrm{d}y} = (u-c)^2 B\int_0^y \frac{\mathrm{d}y_1}{(u_1-c)^2} + By, \quad f_-^{(1)} = By\int_0^y \frac{\mathrm{d}y_1}{(u_1-c)^2}.$$

At $O(k^2)$ the solution at the right is

$$\frac{\mathrm{d}}{\mathrm{d}y}\left[(u-c)^2\frac{\mathrm{d}f_+^{(2)}}{\mathrm{d}y}\right]=\frac{\mathrm{d}}{\mathrm{d}y}\left[(u-c)^2 f_+^{(1)}\right]+(u-c)^2\frac{\mathrm{d}f_+^{(1)}}{\mathrm{d}y},$$

$$(u-c)^2\frac{\mathrm{d}f_+^{(2)}}{\mathrm{d}y}=-(u-c)^2\int_y^\infty \mathrm{d}y_1\left[1-\left(\frac{1-c}{u_1-c}\right)^2\right]-\int_y^\infty \mathrm{d}y_1[(u_1-c)^2-(1-c)^2],$$

$$f_+^{(2)}=\int_y^\infty \mathrm{d}y_1\int_{y_1}^\infty \mathrm{d}y_2\left[\frac{1}{(u_1-c)^2}+\frac{1}{(u_2-c)^2}\right][(u_2-c)^2-(1-c)^2],$$

and the solution at the left is

$$\frac{\mathrm{d}}{\mathrm{d}y}\left[(u-c)^2\frac{\mathrm{d}f_-^{(2)}}{\mathrm{d}y}\right]=\frac{\mathrm{d}}{\mathrm{d}y}\left[(u-c)^2 f_-^{(1)}\right]+(u-c)^2\frac{\mathrm{d}f_-^{(1)}}{\mathrm{d}y},$$

$$(u-c)^2\,\frac{\mathrm{d}f_-^{(2)}}{\mathrm{d}y}=B\left\{y(u-c)^2\int_0^y\frac{\mathrm{d}y_1}{(u_1-c)^2}+\int_0^y \mathrm{d}y_1(u_1-c)^2\int_0^{y_1}\frac{\mathrm{d}y_2}{(u_2-c)^2}+\frac{1}{2}y^2\right\},$$

$$f_-^{(2)}=B\left\{\frac{1}{2}y^2\int_0^y\frac{\mathrm{d}y_1}{(u_1-c)^2}+\int_0^y\frac{\mathrm{d}y_1}{(u_1-c)^2}\int_0^{y_1}\mathrm{d}y_2(u_2-c)^2\int_0^{y_2}\frac{\mathrm{d}y_3}{(u_3-c)^2}\right\}.$$

Thus, accurate to $O(k^3)$ we have the solution at the right of y_N:

$$(u-c)^2\frac{\mathrm{d}f_+}{\mathrm{d}y}=k[(u-c)^2-(1-c)^2]-k^2\int_y^\infty \mathrm{d}y_1[(u-c)^2+(u_1-c)^2]\left[1-\left(\frac{1-c}{u_1-c}\right)^2\right],$$

$$f_+=1-k\int_y^\infty \mathrm{d}y_1\left[1-\left(\frac{1-c}{u_1-c}\right)^2\right]$$
$$+k^2\int_y^\infty \mathrm{d}y_1\int_{y_1}^\infty \mathrm{d}y_2\left[\frac{1}{(u_1-c)^2}+\frac{1}{(u_2-c)^2}\right][(u_2-c)^2-(1-c)^2],$$

and the solution at the left: $f_-=BF_-$,

$$(u-c)^2\frac{\mathrm{d}F_-}{\mathrm{d}y}=1+ky+\frac{1}{2}k^2y^2+k(1+ky)(u-c)^2\int_0^y\frac{\mathrm{d}y_1}{(u_1-c)^2}$$
$$+k^2\int_0^y \mathrm{d}y_1(u_1-c)^2\int_0^{y_1}\frac{\mathrm{d}y_2}{(u_2-c)^2},$$

$$F_-=(1+ky+\frac{1}{2}k^2y^2)\int_0^y\frac{\mathrm{d}y_1}{(u_1-c)^2}+k^2\int_0^y\frac{\mathrm{d}y_1}{(u_1-c)^2}\int_0^{y_1}\mathrm{d}y_2(u_2-c)^2\int_0^{y_2}\frac{\mathrm{d}y_3}{(u_3-c)^2}.$$

The pole $u(y)=c$ in the integrals is indented from below.

It is easy to see that a constant B is determined from the first matching condition in (2.5) and is not involved in the second matching condition. Using the identity

$$\int_0^{y_N}\frac{\mathrm{d}y}{(u-c)^2}\int_0^y \mathrm{d}y_1(u_1-c)^2\int_0^{y_1}\frac{\mathrm{d}y_2}{(u_2-c)^2}$$
$$=\int_0^{y_N}\frac{\mathrm{d}y}{(u-c)^2}\int_0^{y_N}\mathrm{d}y_1(u_1-c)^2\int_0^{y_1}\frac{\mathrm{d}y_2}{(u_2-c)^2}-\int_0^{y_N}\mathrm{d}y(u-c)^2\left[\int_0^y\frac{\mathrm{d}y_1}{(u_1-c)^2}\right]^2,$$

we find

$$
\begin{aligned}
0 &= Jf_+(y_N)F_-(y_N) + \left[(u-c)^2\frac{\mathrm{d}f_+}{\mathrm{d}y}F_- - (u-c)^2\frac{\mathrm{d}F_-}{\mathrm{d}y}f_+\right]_{y=y_N} \\
&= \left[J\int_0^{y_N}\frac{\mathrm{d}y}{(u-c)^2}-1\right]\left[1+ky_N+\frac{1}{2}k^2{y_N}^2+k^2\int_0^{y_N}\mathrm{d}y_1(u_1-c)^2\int_0^{y_1}\frac{\mathrm{d}y_2}{(u_2-c)^2}\right]f_+(y_N) \\
&\quad -k(1+ky_N)\left\{(1-c)^2+k\int_{y_N}^{\infty}\mathrm{d}y_1[(u_1-c)^2-(1-c)^2]\right\}\int_0^{y_N}\frac{\mathrm{d}y_1}{(u_1-c)^2} \\
&\quad -Jk^2\int_0^{y_N}\mathrm{d}y(u-c)^2\left[\int_0^{y}\frac{\mathrm{d}y_1}{(u_1-c)^2}\right]^2+O(k^3). \qquad \text{(A 1)}
\end{aligned}
$$

With the same accuracy, this equation can be cancelled by $(1+ky_N+\frac{1}{2}k^2{y_N}^2)$; after that, on excluding the terms of $O(k^3)$ when $|u_N-c|\gg k$, and $O(k^2)$ when $|u_N-c|=O(k)$, we obtain the dispersion equation (3.1).

For a weak stratification, $J=O(k)$, the difference $|u_N-c|=O(k)$, and integrals in (A1), having a resonance denominator, become of $O(k^{-1})$ (with the exception of the last one). We put

$$
\int_0^{y_N}\frac{\mathrm{d}y}{(u-c)^2}=\frac{1}{u'_N(c-u_N)}-\frac{u''_N}{u'^3_N}\ln\frac{(c-u_N)u'_N}{k}+i_1,
$$

$$
\int_{y_N}^{\infty}\mathrm{d}y_1\left[1-\left(\frac{1-c}{u_1-c}\right)^2\right]=(1-c)^2\left[\frac{1}{u'_N(c-u_N)}-\frac{u''_N}{u'^3_N}\ln\frac{(c-u_N)u'_N}{k}+\frac{\mathrm{i}\pi u''_N}{u'^3_N}\right]+i_2,
$$

$$
\int_{y_N}^{\infty}\frac{\mathrm{d}y_1}{(u_1-c)^2}\int_{y_1}^{\infty}\mathrm{d}y_2[(u_2-c)^2-(1-c)^2]=\frac{i_N}{u'_N(c-u_N)}+O\left(\ln\frac{c-u_N}{k}\right),
$$

where

$$
i_N=\int_{y_N}^{\infty}\mathrm{d}y[(1-c)^2-(u-c)^2],
$$

and i_1 and i_2 are $O(1)$. With the notation $z=u'_N(c-u_N)/k$, and $J=k\tilde{J}$, from (A 1) we obtain the dispersion equation

$$
\begin{aligned}
z^2-\tilde{J}z+\tilde{J}(1-u_N)^2 &= \tilde{J}k\left\{-\mathrm{i}\pi\frac{u''_N}{u'^3_N}(1-u_N)^4+\frac{u''_N}{u'^3_N}[2(1-u_N)^2-z]z\ln z\right. \\
&\quad \left.+2\frac{1-u_N}{u'_N}z+z(1-u_N-z)^2i_1-(1-u_N)^2i_2+i_N\right\}+O(k^2\ln k). \qquad \text{(A 2)}
\end{aligned}
$$

Appendix B. Oscillations in the neighbourhood of neutral modes

B.1. *Oscillations in the vicinity of the neutral mode with $c=1$*

When $c\to 1-0$, the resonance point $y_c\to+\infty$ and for this reason the details of calculations depend on the asymptotic behaviour of $u(y)$. For definiteness, we assume that

$$
u(y)\approx 1-v\mathrm{e}^{-\alpha y};\quad v>0,\quad \alpha>0 \qquad \text{(B 1)}
$$

when $y\to\infty$. We put $c=1-vs$ and $|s|\ll 1$, and use in (2.2) the variable $z=\mathrm{e}^{-\alpha y}$:

$$
\left.
\begin{aligned}
&z\frac{\mathrm{d}}{\mathrm{d}z}\left(z\frac{\mathrm{d}g}{\mathrm{d}z}\right)+\left(\frac{u''}{1-u-vs}-k^2\right)\frac{g}{\alpha^2}=0;\quad |g(0)|<\infty,\quad g(1)=0; \\
&g_+(z_N)=g_-(z_N),\quad z_N\left(\frac{\mathrm{d}g_+}{\mathrm{d}z}-\frac{\mathrm{d}g_-}{\mathrm{d}z}\right)_{z=z_N}+\frac{J\,g(z_N)}{\alpha(1-u_N-vs)^2}=0,
\end{aligned}
\right\} \qquad \text{(B 2)}
$$

where $u'' = \mathrm{d}^2u/\mathrm{d}y^2$ is regarded as the function of z, and g_+ and g_- are the solutions for $z > z_N$ and $z < z_N$, respectively.

For $c = 1$ ($s = 0$) the general solution of equation (B 2) in the limit $z \to 0$ has an asymptotic representation (if $2q$ is not equal to an integer):

$$g = A_1 z^q[1 + O(z)] + A_2 z^{-q}[1 + O(z)]; \quad q = (1 + k^2/\alpha^2)^{1/2}.$$

Let us denote the bounded solution ($A_1 = 1$, $A_2 = 0$) by $G(z)$ and express the solution of the problem (B 2) in terms of it. Up to an arbitrary common factor,

$$g \equiv g_0(z) = \begin{cases} g_-(z) = G(z), & z < z_N, \\ g_+(z) = BF(z), & z > z_N, \end{cases} \tag{B 3}$$

where

$$F(z) = G(z)\int_z^1 \frac{\mathrm{d}z_1}{z_1 G^2(z_1)}, \quad B = \frac{1}{\alpha I_0} = \left[\int_{z_N}^1 \frac{\mathrm{d}z}{zG^2(z)}\right]^{-1};$$

$$J = J_1(k) = \alpha(1 - u_N)^2\left[G^2(z_N)\int_{z_N}^1 \frac{\mathrm{d}z}{zG^2(z)}\right]^{-1}.$$

When $s \neq 0$, equation (B 2) has, along with $z = 0$, a singular point $1 - u - vs = 0$, or $z \approx s$. We construct the solution using the method of matched asymptotic expansions.

In the 'outer' region ($z \gg |s|$) we seek the solution in the form of an expansion (formally, in the parameter s):

$$g_\pm = g_\pm{}^{(0)} + g_\pm{}^{(1)} + \ldots, \quad J = J_1(k) + J^{(1)}(k) + \ldots,$$

where $g_\pm^{(0)}$ are determined by formulae (B 3),

$$g_+^{(1)} = \frac{Bvs}{\alpha^2}\int_z^1 \frac{\mathrm{d}z_1 u''(z_1)}{z_1[1 - u(z_1)]^2}F(z_1)[F(z_1)G(z) - F(z)G(z_1)],$$

$$g_-^{(1)} = \frac{vs}{\alpha^2}\int_z^{z_N} \frac{\mathrm{d}z_1 u''(z_1)}{z_1[1 - u(z_1)]^2}G(z_1)[F(z_1)G(z) - F(z)G(z_1)] + B_1BF(z) + B_2G(z).$$

Matching $g_+^{(1)}$ and $g_-^{(1)}$ at $z = z_N$ gives

$$B_1 = -\frac{J^{(1)}}{J_1} - \frac{2vs}{1 - u_N} - \frac{Bvs}{\alpha^2}\int_{z_N}^1 \frac{\mathrm{d}z\, u''}{z(1 - u)^2}F^2,$$

$$B_2 = \frac{J^{(1)}}{J_1} + \frac{2vs}{1 - u_N} + \frac{vs}{\alpha^2}\int_{z_N}^1 \frac{\mathrm{d}z\, u''}{z(1 - u)^2}F(2BF - G).$$

An asymptotic representation of the solution for $z \to 0$ has the form (only the main terms are written out):

$$g(z) = z^q - \frac{z^{-q}}{2q}\left[\frac{BJ^{(1)}}{J_1} + \frac{2Bvs}{1 - u_N} + \frac{vs}{\alpha^2}\int_0^1 \frac{\mathrm{d}z u''}{z(1 - u)^2}g_0^2(z)\right] + \ldots. \tag{B 4}$$

In the 'inner' region ($z = O(|s|)$) we introduce the variable $x = z/s$ and, in view of (B 1), from (B 2) we obtain

$$x\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}g}{\mathrm{d}x}\right) + \left(\frac{x}{1 - x} - \frac{k^2}{\alpha^2}\right)g = 0, \quad |g(0)| < \infty.$$

Substitution of $g = x^\beta h(x)$ with $\beta = k/\alpha$ leads to a hypergeometric equation

$$x(1 - x)\frac{\mathrm{d}^2h}{\mathrm{d}x^2} + (2\beta + 1)(1 - x)\frac{\mathrm{d}h}{\mathrm{d}x} + h = 0,$$

the bounded (when $x \to 0$) solution of which has the form

$$h = AF(\beta + q, \beta - q; 2\beta + 1; x), \quad A = \text{const},$$

where $F(a, b; c; z)$ is a hypergeometric function (Erdelyi 1953). Using the formulae for analytic continuation of $F(a, b; c; z)$ (Erdelyi 1953) we find the asymptotic representation of $g(z)$ when $z \gg |s|$ (i.e. $x \gg 1$), provided that $|\arg(-s)| < \pi$:

$$g(z) = A\frac{\Gamma(2\beta+1)\Gamma(2q)}{(\beta+q)[\Gamma(\beta+q)]^2}\frac{(-s)^{\beta-q}}{s^{\beta}}\left\{z^q - \frac{\Gamma(-2q)}{\Gamma(2q)}\left[\frac{\Gamma(\beta+q)}{\Gamma(\beta-q+1)}\right]^2\frac{(-s)^{2q}}{z^q} + \dots\right\}.$$

Here $\Gamma(x)$ is the Euler gamma function (Erdelyi 1953).

Matching with (B 4) yields the dispersion equation

$$\frac{J^{(1)}}{J_1} + vsB_0 + \frac{\pi\alpha I_0(-s)^{2q}}{\sin 2\pi q}\left[\frac{\Gamma(\beta+q)}{\Gamma(2q)\Gamma(\beta-q+1)}\right]^2 = 0, \tag{B 5}$$

where

$$B_0 = \frac{2}{1-u_N} + \frac{I_0}{\alpha}\int_0^1 \frac{\mathrm{d}z u''}{z(1-u)^2}g_0^2(z) \equiv \frac{2}{1-u_N} + I_0\int_0^{\infty}\frac{\mathrm{d}y u''}{(1-u)^2}g_0^2(y) > 0.$$

Since $q > 1$, the last term in this equation is small compared to the second term, but it can carry information about the imaginary part of c. We solve (B 5) by the method of successive approximations. At the main order

$$J^{(1)} = -vsB_0J_1. \tag{B 6}$$

Multiplying equation (B 2) with $s = 0$ by $2g_0/[z(1-u)]$ and integrating over z from 0 to 1, in view of the boundary conditions and matching conditions, it is easy to show that $B_0 > 0$, so that $J^{(1)}$ and s have opposite signs.

If $J^{(1)} > 0$ (i.e. $J > J_1(k)$), then $(-s) > 0$, and the last term in (B 5) gives only a small real correction to (B 6) – the flow is stable, as would be expected. If, however, $J^{(1)} < 0$, then to a first approximation (see (B 6)) $s > 0$, and $(-s)^{2q}$ is a complex number. In view of the limitation $|\arg(-s)| < \pi$, the equality $(-s) = s\mathrm{e}^{\mathrm{i}\pi}$ corresponds to the unstable mode ($\mathrm{Im}\, c > 0$). Upon substituting into (B 5), we find that when $J < J_1(k)$

$$c = 1 - vs \approx 1 - \frac{J_1 - J}{B_0J_1} + \frac{\mathrm{i}\pi\alpha I_0}{B_0}\left[\frac{\Gamma(q + k/\alpha)}{\Gamma(2q)\Gamma(1 + k/\alpha - q)}\right]^2\left(\frac{J_1 - J}{vB_0J_1}\right)^{2q}.$$

B.2. *Oscillations in the vicinity of the neutral mode with $c = c_n$*

Let $g(y)$ be an eigenfunction of the neutral mode with $c = c_n$. Using the designations

$$\hat{L} = \frac{\mathrm{d}^2}{\mathrm{d}y^2} + \frac{N^2(y)}{(u-c_n)^2} - \frac{u''}{u-c_n} - k^2; \quad h_0 = \frac{\partial g}{\partial k^2}, \quad h_1 = \frac{\partial g}{\partial J}, \quad h_2 = \frac{\partial^2 g}{\partial J^2},$$

from (2.1), in view of (1.1), we obtain

$$\left.\begin{aligned}\hat{L}h_0 &= g + \left[\frac{u''}{(u-c_n)^2} - \frac{2N^2}{(u-c_n)^3}\right]\frac{\partial c}{\partial k^2}g,\\ \hat{L}h_1 &= -\frac{\delta(y-y_N)}{(u-c_n)^2}\,g + \left[\frac{u''}{(u-c_n)^2} - \frac{2N^2}{(u-c_n)^3}\right]\frac{\partial c}{\partial J}g,\\ \hat{L}h_2 &= -\frac{2\delta(y-y_N)}{(u-c_n)^2}\left(h_1 + \frac{2g}{u-c_n}\frac{\partial c}{\partial J}\right) + \left[\frac{u''}{(u-c_n)^2} - \frac{2N^2}{(u-c_n)^3}\right]\\ &\quad\times\left(2\frac{\partial c}{\partial J}h_1 + \frac{\partial^2 c}{\partial J^2}g\right) + 2\left[\frac{u''}{(u-c_n)^3} - \frac{3N^2}{(u-c_n)^4}\right]\left(\frac{\partial c}{\partial J}\right)^2 g.\end{aligned}\right\} \tag{B 7}$$

Multiplying equations (B 7) by g and integrating over y, we find

$$\left.\begin{aligned}D(k)\frac{\partial c}{\partial k^2} &= -\int_0^\infty \mathrm{d}y\, g^2(y),\\ D(k)\frac{\partial c}{\partial J} &= \frac{{g_N}^2}{(u_N-c_n)^2},\\ D(k)\frac{\partial^2 c}{\partial J^2} &= \frac{2g_N h_{1N}}{(u_N-c_n)^2} - 2\frac{\partial c}{\partial J}\left[\int_0^\infty \frac{\mathrm{d}y\, u'' g\, h_1}{(u-c_n)^2} - \frac{2g_N(g_N + J\,h_{1N})}{(u_N-c_n)^3}\right]\\ &\quad -2\left(\frac{\partial c}{\partial J}\right)^2\left[\int_0^\infty \frac{\mathrm{d}y\, u'' g^2}{(u-c_n)^3} - \frac{3J{g_N}^2}{(u_N-c_n)^4}\right],\end{aligned}\right\} \tag{B 8}$$

where $g_N = g(y_N)$, $h_{1N} = h_1(y_N)$,

$$D(k) = \int_0^\infty \frac{\mathrm{d}y\, u'' g^2}{(u-c_n)^2} + \frac{2J{g_N}^2}{(c_n-u_N)^3}. \tag{B 9}$$

From the first two equations in (B 8) it is easy to see that the first derivatives of c with respect to k^2 and J are real and $(\mathrm{d}J/\mathrm{d}k^2)_{c=c_n} > 0$, so that $J^{(n)}(k)$ is a monotonically increasing function on each branch of the neutral curve.

Further, it is obvious that $h_1(y)$ is a real function, so that an imaginary contribution to the second derivative of c with respect to J is made only by the last integral in (B 8):

$$D(k)\,\mathrm{Im}\,\frac{\partial^2 c}{\partial J^2} = -\frac{\pi u_c^{\mathrm{iv}}}{{u_c}'^3}{g_c}^2\left(\frac{\partial c}{\partial J}\right)^2. \tag{B 10}$$

Upon substituting into the Taylor expansion for c, we obtain (4.3).

To describe the transition over the neutral curve at the critical points k_* and k_+, we put $J = J(c)$ and, as earlier, we obtain (cf. (B 8)):

$$\frac{{g_N}^2}{(u_N-c_n)^2}\frac{\partial J}{\partial c} = D(k) = 0,$$

$$\frac{{g_N}^2}{(u_N-c_n)^2}\frac{\partial^2 J}{\partial c^2} = 2\int_0^\infty \frac{\mathrm{d}y\, u'' g}{(u-c_n)^2}\left(h_1 + \frac{g}{u-c_n}\right) + \frac{2J\,g_N}{(c_n-u_N)^3}\left(2h_{1N} - \frac{3g_N}{c_n-u_N}\right).$$

From this, by analogy with (B 10), we find

$$\mathrm{Im}\,\frac{\partial^2 J}{\partial c^2} = \frac{\pi u_c^{\mathrm{iv}}}{{u_c}'^3}(u_N-c_n)^2\left(\frac{g_c}{g_N}\right)^2 < 0.$$

Thus near the neutral curve in this case

$$J - J^{(n)} \approx \frac{1}{2}\frac{\partial^2 J}{\partial c^2}(c - c_n)^2, \qquad c - c_n \sim \left(J - J^{(n)}\right)^{1/2}. \tag{B 11}$$

Now we investigate the sign of $D(k)$ on the neutral curve $J = J^{(n)}(k)$. The first term in (B 9) is always negative and the second term is positive only when $u_N < c_n$. In this case the neutral curve has two branches. At the left end of the long-wavelength branch, when $k = 0$, we have (see (3.2) and Appendix A):

$$g_+(y) = u - c_n, \quad g_-(y) = J\,(u - c_n)\int_0^y \frac{\mathrm{d}y_1}{(u_1 - c_n)^2}, \quad J = J^{(n)}(0) = \left[\int_0^{y_N} \frac{\mathrm{d}y}{(u - c_n)^2}\right]^{-1}.$$

In (B 9) we represent $\int_0^\infty \mathrm{d}y \ldots = \int_0^{y_N} \mathrm{d}y \ldots + \int_{y_N}^\infty \mathrm{d}y \ldots$, and after integrating twice by parts we find

$$D(0) = J^2 \int_0^{y_N} \mathrm{d}y\, u'' \left[\int_0^y \frac{\mathrm{d}y_1}{(u_1 - c_n)^2}\right]^2 - u_N{}' + \frac{2J}{c_n - u_N} = 2J^2 \int_0^{y_N} \frac{\mathrm{d}y}{(c_n - u)^3} > 0.$$

Let us now consider the right end of this same branch, $k \to k_G - 0$, which is determined by the fact that $y_G \to y_N + 0$. Since $G(y_G) = 0$, for $|y - y_G| \ll 1$ we put $G(y) \approx \alpha(y - y_G)$ and, using (2.4) and (4.1), obtain

$$g_N = G_N \approx -\alpha(y_G - y_N), \qquad F_N \approx -\alpha^{-1}, \qquad J^{(n)}(k) \approx \frac{(u_N - c_n)^2}{y_G - y_N},$$

so that the second term in (B 9) tends to zero as $(y_G - y_N)$. Consequently, at this end $D(k) < 0$. Turning next to the short-wavelength branch, at its left end when $k \to k_n + 0$, we see that $J \to 0$ and $D(k) < 0$. At the right end, however, when $k \gg 1$, $g_\pm \approx g_N \exp(-k|y - y_N|)$, $J^{(n)}(k) \approx 2\,(c_n - u_N)^2 k + J_*{}^{(n)}$, and

$$D(k) \approx g_N{}^2 \left[\frac{u_N{}''}{k(u_N - c_n)^2} + \frac{4(c_n - u_N)^2 k + 2J_*{}^{(n)}}{(c_n - u_N)^3}\right] = \frac{4\,k\,g_N{}^2}{c_n - u_N} + O(1) > 0.$$

Thus, if $c_n > u_N$, at each of the neutral curve branches, $D(k)$ changes its sign on going from its left to right end: on the long-wavelength branch from plus to minus when $k = k_*$, and on the short-wavelength branch from minus to plus when $k = k_+$ (see figure 3*a*). If, however, $c_n < u_N$, then $D(k) < 0$ throughout the neutral curve (figure 3*b, c*).

Appendix C. Properties of $F(y)$ and $G(y)$ in flows with an NCP

In a homogeneous flow with $u''(y)$ of fixed sign (strictly negative, in particular) there are no eigen-oscillations. Mathematically, this implies that when $k > 0$ the functions $F(y)$ and $G(y)$ introduced in (2.4) have no real zeros (except for $F(0) = 0$) for any complex c. In flows with an NCP, eigen-oscillations do appear: to each NCP $y = y_n$ there corresponds a neutral oscillation with the phase velocity $c = c_n = u(y_n)$.

Indeed, when $c = c_n$, the Rayleigh equation has no singularity at the point of phase resonance $u(y) = c$, and the problem of eigen-oscillations is equivalent to the problem of energy levels $E = -k^2$ of a quantum particle in the potential well of finite depth $U(y) = u''/(u - c_n)$ (see figure 9) with the infinitely high wall at $y = 0$. The number of levels in the well is equal to the number of zeros of the bounded (when $y \to \infty$) solution of the Schrödinger equation with $E = 0$ (Calogero & Degasperis 1982). For the Rayleigh equation such a solution is represented by the function $G(y)$ which when

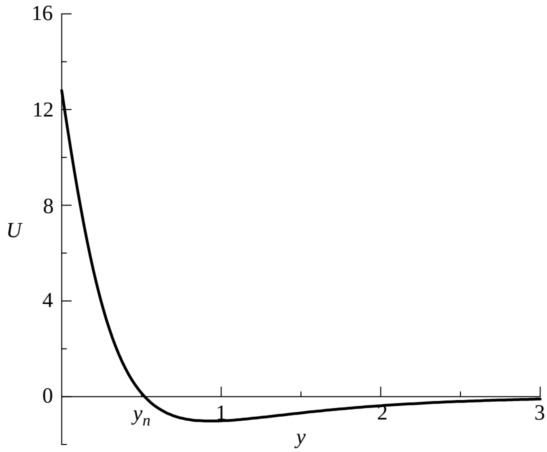

FIGURE 9. Potential $U(y)$ for the eigenvalue problem in the homogeneous flow (4.8).

$k = 0$ is equal to $(u - c_n)$, and, because of the monotonic character of $u(y)$, it has exactly one zero, $y = y_n$. When $k > 0$, $G(y)$ remains real along the axis y, and its zero, $y = y_G$, with increasing k, shifts to the left to reach $y = 0$ at a certain $k = k_n > 0$: k_n and the corresponding $G(y)$ are just the wavenumber and the eigenfunction of the eigen-oscillation, related to the NCP $y = y_n$. Note that when $k = k_n$, not only $g_+ = G$ but also $g_- = G$, each one in its domain of definition.

The function $F(y)$, when $c = c_n$, is also real along the axis y. When $k = 0$ it is negative (together with $G(y)$) in the region $0 < y < y_n$, and positive when $y \to \infty$. Consequently, it has zero at $y = y_{F0} > y_n$ that is determined by the equation†

$$\int_0^{y_{F0}} \frac{\mathrm{d}y}{(u - c_n)^2} = 0.$$

With an increase of k, the zero of $F(y)$, $y = y_F$, shifts along the axis y to the right, and when $k \to k_n$ it goes to infinity. Thus if $k < k_n$, $F(y)$ is negative when $0 < y < y_F$ and positive when $y > y_F$, and $G(y)$ is negative when $y < y_G \leqslant y_n < y_F$ and positive when $y > y_G$. If, however, $k > k_n$, the two functions are positive everywhere on $0 < y < \infty$.

Appendix D. Analysing the roots of equations (5.3) and (5.5)

The cubic (for c) equations (5.3) and (5.5) should be written first in a reduced form

$$s^3 + ps + q = 0, \tag{D 1}$$

and then the solutions of (D 1) may be written using the Cardano formulae (see, for example, Korn & Korn 1968):

$$\left. \begin{aligned} s_1 = w_+ + w_-, \quad s_{2,3} &= -\frac{w_+ + w_-}{2} \pm \frac{\sqrt{3}}{2}\,\mathrm{i}\,(w_+ - w_-); \\ w_\pm = \left(-\frac{q}{2} \pm D^{1/2}\right)^{1/3}, \quad D &= \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2. \end{aligned} \right\} \tag{D 2}$$

† The pole $y = y_n$ should be indented in a complex plane.

In the case of real p and q, equation (D1) has a pair of complex conjugate roots if $D > 0$, whereas all roots are real when $D \leqslant 0$.

1. $\boldsymbol{y_N \geqslant 1}$**.** In equation (5.3) we assume $s = c - 1 + t/[3k(1+t)]$; then

$$p = -\left[\frac{t^2}{3k^2(1+t)^2} + \frac{J}{2k}\left(1 - \frac{1-t}{1+t}E\right)\right],$$

$$q = \frac{2t^3}{27k^3(1+t)^3} - \frac{Jt}{3k^2(1+t)}\left(1 - \frac{1+2t}{1+t}E\right),$$

$$D = -\frac{Jt^4}{54k^5(1+t)^4}\left\{\frac{1}{4}\left(1 - \frac{1-t}{1+t}E\right)^3\left[\frac{Jk(1+t)^2}{t^2}\right]^2 - S\frac{Jk(1+t)^2}{t^2} + 1 - E\right\},$$

$$S = 1 - \frac{2+7t}{1+t}E + \frac{1+7t+\frac{11}{2}t^2}{(1+t)^2}E^2.$$

Instability is possible in the case where the quadratic trinomial (relative to $Jk(1+t)^2/t^2$) involved in D in curly brackets is negative. Its roots

$$\left.\begin{aligned} &\left[\frac{Jk(1+t)^2}{t^2}\right]_{\pm} = 2\left(1 - \frac{1-t}{1+t}E\right)^{-3}\left(S \pm \varDelta^{1/2}\right), \\ &\varDelta = S^2 - (1-E)\left(1 - \frac{1-t}{1+t}E\right)^3 = \frac{16Et}{1+t}\left[\frac{5t+4}{4(1+t)}E - 1\right]^3, \end{aligned}\right\} \qquad \text{(D 3)}$$

are real and different if $E > 4(1+t)/(4+5t)$. We fix k (and $t = \tanh k$) and seek the region of parameters in which inequality holds. Since $E = \mathrm{e}^{-2k(y_N-1)} \leqslant 1$, both roots (D 3) have the same sign, coinciding with the sign of S. A minimum S (in E) is reached when

$$E = \frac{(1+t)(2+7t)}{2+14t+11t^2} < \frac{4+4t}{4+5t},$$

i.e. the desired region lies on the increasing branch of the parabola $S(E)$. Therefore in this region

$$S(E) \leqslant S(1) = 1 - \frac{2+7t}{1+t} + \frac{1+7t+\frac{11}{2}t^2}{(1+t)^2} = -\frac{t^2}{2(1+t)^2} < 0,$$

so that both roots (D 3) can only be negative, and this is possible only for unstable stratification ($J < 0$). Thus, for stable stratification, all roots of (5.3) are real, i.e. the flow (1.2) with $y_N \geqslant 1$ is stable.

2. $\boldsymbol{y_N < 1}$**.** In equation (5.5) it is assumed (for designations, see the main text) that $s = c - u_N - d_1/(6k)$ where $d_1 = d + (1-t)E/(1+t)$, then

$$p = -\left[\tilde{J} + \frac{{d_1}^2}{12k^2}\right], \qquad q = \frac{\tilde{J}}{3k}\left(d + \frac{1+2t}{1+t}E\right) - \frac{{d_1}^3}{108k^3},$$

$$D = -\frac{\tilde{J}}{27}\left[\tilde{J}^2 - 2S\tilde{J} + \frac{{d_1}^3}{(2k)^4}(d+E)\right], \qquad \tilde{J} = \frac{Jt}{k(1+t)};$$

$$S = \frac{1}{4k^2}\left[d^2 + \frac{2+7t}{1+t}Ed + \frac{1+7t+\frac{11}{2}t^2}{(1+t)^2}E^2\right] = \frac{d_1}{4k^2}\left(d + \frac{1+8t}{1+t}E\right) + \frac{27t^2E^2}{8k^2(1+t)^2}.$$

It is easy to see that when $k > 0$

$$d + E = -[1 - 2k(1-u_N)] + \exp[-2k(1-u_N)] > 0,$$

so that the discriminant of the quadratic trinomial involved in the expression for D

$$\Delta = S^2 - \frac{{d_1}^3}{(2k)^4}(d+E) = \frac{Et}{(1+t)k^4}\left[d + \frac{4+5t}{4(1+t)}E\right]^3 > 0,$$

and D becomes zero when $\tilde{J} = 0$ or $\tilde{J} = \tilde{J}_{\pm} = S \pm \Delta^{1/2}$. If $d_1 > 0$, then $\tilde{J}_{\pm} > 0$ (since $S > 0$), otherwise $\tilde{J}_{+} > 0$ and $\tilde{J}_{-} < 0$. $\tilde{J}_{-}$ becomes zero if $d_1 = 0$, from which follows (5.7).

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